

# Distinction of representations via Bruhat-Tits buildings of $p$ -adic groups

À la mémoire de mon ami François Courtès 1970–2016

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## Introduction

Let  $G/H$  be a symmetric space over a non-archimedean local field  $F$ :  $G$  is (the group of  $F$ -points of) a reductive group over  $F$  and  $H \subset G$  is the subgroup of ( $F$ -rational) points in  $G$  fixed by an involution. A local counterpart of the theory of periods of automorphic forms on adèle groups is the harmonic analysis on the coset space  $G/H$ . The irreducible complex representations  $\pi$  of  $G$  which contribute to harmonic analysis on  $G/H$  are those representations  $\pi$  which embed in the induced representation  $\text{Ind}_H^G \mathbb{C}$ , where  $\mathbb{C}$  denotes the trivial character of  $H$ . By Frobenius reciprocity this amounts to asking that the

intertwining space  $\text{Hom}_G(\pi, \mathbb{C})$  is non zero. Such representations are called distinguished by  $H$ . If  $\pi$  is distinguished, a non zero linear form  $\Lambda \in \text{Hom}_G(\pi, \mathbb{C})$  is sometimes called a local period for  $\pi$  relative to  $H$ .

Among symmetric spaces one has the family of Galois symmetric spaces, that is quotients of the form  $\mathbb{G}(E)/\mathbb{G}(F)$ , where  $E/F$  is a Galois quadratic extension of  $p$ -adic fields and  $\mathbb{G}$  is a reductive group over  $F$ . By the conjectural local Langlands correspondence an irreducible representation  $\pi$  of  $\mathbb{G}(E)$  possesses a Galois parameter  $\varphi_\pi$ . In [Pr2] Dipendra Prasad proposes a “relative local Langlands correspondence” of conjectural nature: he gives a conjectural list of conditions on the parameter  $\varphi_\pi$  in order that  $\pi$  be distinguished by  $\mathbb{G}(F)$ .

Among the irreducible representations of  $p$ -adic reductive groups, one is somehow “universal”; this is the Steinberg representation. Its definition is uniform and it has nice models of geometric nature. It is therefore natural to test Prasad’s conjecture with this particular representation. In fact in the earlier paper [Pr], Prasad gave a conjecture on the Steinberg representation which turns out to be a particular case of the previous conjecture.

Let  $\mathbb{G}(E)/\mathbb{G}(F)$  be a Galois symmetric space and assume that  $\mathbb{G}$  is quasi-split over  $F$ . In [Pr] Prasad defines a quadratic character  $\epsilon$  of  $\mathbb{G}(F)$  and makes the following conjecture.

**Conjecture** ([Pr] Conjecture 3, p. 77). *Let  $\mathbf{St}_E$  be the Steinberg representation of  $\mathbb{G}(E)$ .*

- (a) *The intertwining space  $\text{Hom}_{\mathbb{G}(F)}(\mathbf{St}_E, \epsilon)$  is 1-dimensional.*
- (b) *If  $\chi \neq \epsilon$  is any other character of  $\mathbb{G}(F)$ , then  $\text{Hom}_{\mathbb{G}(F)}(\mathbf{St}_E, \chi) = 0$ .*

In [BC] the author and F. Courtès gave a proof of Prasad’s conjecture when  $\mathbb{G}$  is split over  $F$  and  $E/F$  is unramified (actually there were some other conditions on the group  $\mathbb{G}$  and on the size of the residue field of  $F$ , but they were removed later). The aim of this expository work is to explain some of the ideas used in the proof given in [BC].

Let  $G$  be a reductive group over a  $p$ -adic field. The approach of [BC] is based on the model of the Steinberg representation of  $G$  given by the cohomology of its Bruhat-Tits building  $X_G$ . As a topological space,  $X_G$  is a locally compact space on which  $G$  acts properly (mod center). It is a result of A. Borel and J.-P. Serre [BS] that as a  $G$ -module the top cohomology space with compact support  $H_c^{\text{top}}(X_G, \mathbb{C})$  is an irreducible smooth representation of  $G$  isomorphic to the Steinberg representation  $\mathbf{St}_G$ . From this result it is easy to construct a model of the Steinberg representation as a subspace of the space of complex functions on the set of chambers of  $X_G$ . Indeed let  $\mathcal{H}(X_G)$  be the space of harmonic functions on chambers of  $X_G$ , that is complex functions  $f$  satisfying

$$\sum_{C \supset D} f(C) = 0$$

for all codimension 1 simplex  $D$  of  $X_G$ . Then one has natural isomorphisms of  $G$ -modules:

$$\text{Hom}_{\mathbb{C}}(\mathbf{St}_G, \mathbb{C}) \simeq \mathcal{H}(X_G) \otimes \epsilon', \quad \mathbf{St}_G \simeq \mathcal{H}(X_G)^\infty \otimes \epsilon'$$

where  $\mathcal{H}(X_G)^\infty$  denotes the space of smooth vectors in the  $G$ -module  $\mathcal{H}(X_G)$ ; and where  $\epsilon'$  is a certain character of  $G$ .

In the case of a Galois symmetric space  $\mathbb{G}(E)/\mathbb{G}(F)$  satisfying the hypothesis of [BC], a non-zero equivariant linear form  $\Lambda \in \text{Hom}_{\mathbb{G}(F)}(\mathbf{St}_E, \epsilon)$  is given by

$$\Lambda(f) = \sum_{C \subset X_F} f(C), \quad f \in \mathbf{St}_E \simeq \mathcal{H}(X_E)^\infty$$

where the sum is over those chambers of  $X_E := X_{\mathbb{G}(E)}$  which lie in  $X_F := X_{\mathbb{G}(F)}$  (the building  $X_F$  embeds in  $X_E$  canonically).

Section 4 of this article will be devoted to the proof of the fact that the sum above converges, for all  $f \in \mathcal{H}(X_E)^\infty$ , to define a non-zero linear form. Our approach here will be different from that of [BC]; it is based on a new ingredient, namely the Poincaré series of an affine Weyl group, that did not appear in [BC].

We also take the opportunity to give an introductory and pedagogical treatment of the technical background of §4. Namely we start with a review of the theory of Bruhat-Tits building (section 1), then we state the Borel-Serre theorem and give an idea of its proof (section 2). As an exercise we give a complete proof in the case of  $\text{GL}(2)$ . Section 3 is devoted to the Steinberg representation. We review its equivalent definitions and its various models.

Originally this article was part of a bigger project joint with François Courtès. Unfortunately François passed away in septembre 2016 and I resigned myself to writing on my contribution only. However I shall say a few words on François's contribution in §4.5.

Throughout this article we shall use the following notation. The symbol  $F$  will denote a non-archimedean, non-discrete, locally compact field. We fix a prime number  $p$  and assume that  $F$  is either a finite extension of the field  $\mathbb{Q}_p$  of  $p$ -adic numbers or a field  $\mathbb{F}_q((X))$  of Laurent series over a finite field  $\mathbb{F}_q$  of  $q$  elements, where  $q$  is a power of  $p$ . For an introduction to such topological fields, the reader may read chapter I of [W], or [Go]. We shall say that  $F$  is a  *$p$ -adic field*. To any  $p$ -adic field  $K$ , we attach: its normalized valuation  $v_K : K \longrightarrow \mathbb{Z} \cup \{+\infty\}$  (assumed to be onto), its valuation ring  $\mathfrak{o}_K = \{x \in K; v_K(x) \geq 0\}$ , its valuation ideal  $\mathfrak{p}_K = \{x \in K; v_K(x) > 0\}$  and its residue field  $\mathbb{F}_K = \mathfrak{o}_K/\mathfrak{p}_K$ , a finite extension of  $\mathbb{F}_p$ . The cardinal of  $\mathbb{F}_K$  is denoted by  $q_K$ . We fix a quadratic separable extension  $E/F$ . Two cases may occur: either  $\mathfrak{p}_F \mathfrak{o}_E = \mathfrak{p}_E$  (the extension is *unramified*), or  $\mathfrak{p}_F \mathfrak{o}_E = \mathfrak{p}_E^2$  (the extension is *ramified*). We shall work under the following assumption:

(A1) *When  $E/F$  is ramified, the prime number  $p$  is not 2.*

In other words, we assume that the extension  $E/F$  is tame (cf. [Fr] §8).

We fix a connected reductive algebraic group  $\mathbb{G}$  defined over  $F$ . We shall always assume:

(A2) *The reductive group  $\mathbb{G}$  is split over  $F$ .*

For simplicity sake, we also assume the following, even though our results hold without this assumption:

(A3) *The root system of  $\mathbb{G}$  is irreducible.*

Prasad's conjecture deal with the symmetric space obtained from the reductive group  $\mathbb{H} = \text{Res}_{E/F}\mathbb{G}$  (restriction of scalars). If  $\bar{F}$  denotes an algebraic closure of  $F$ , we have an isomorphism of  $\bar{F}$ -algebraic groups:  $\mathbb{H}(\bar{F}) \simeq \mathbb{G}(\bar{F}) \times \mathbb{G}(\bar{F})$ . Let  $\sigma$  be the  $F$ -rational involution of  $\mathbb{H}$  given by  $\sigma(g_1, g_2) = (g_2, g_1)$ ; we have  $\mathbb{H}^\sigma = \mathbb{G}$  (fixed point set). Set  $G_F = \mathbb{G}(F)$  and  $G_E = \mathbb{G}(E)$ . Then  $\mathbb{H}(F) = G_E$  and the action of  $\sigma$  on  $\mathbb{H}(F)$  corresponds to the action of the non-trivial element of the Galois group  $\text{Gal}(E/F)$  on  $G_E$ ; this action will be also denoted by  $\sigma$ . So viewed as a group quotient, the symmetric space attached to the group  $\mathbb{H} = \text{Res}_{E/F}\mathbb{G}$  equipped with the involution  $\sigma$  is  $G_E/G_F$ ; this is what we called a Galois symmetric space.

## 1 The Bruhat-Tits building

**1.1 Apartments and simplicial structure** For an introduction to the concept of building the reader read the monography [AB]. Basic ideas and various applications of this theory are described in [Ro1] and [Ro2].

To any reductive group  $\mathbb{H}$  defined over a  $p$ -adic field  $K$ , the Bruhat-Tits theory ([BT], [BT2]) attaches a (*semisimple*, or *non-enlarged*) building  $\text{BT}(\mathbb{H}, K)$  equipped with an action of  $\mathbb{H}(K)$ . In the sequel we abbreviate  $H = \mathbb{H}(K)$  and  $X_H = \text{BT}(\mathbb{H}, K)$ . Moreover to make things simpler we assume  $\mathbb{H}$  is split over  $F$  and that if  $\mathbb{Z}$  denotes the connected center of  $\mathbb{H}$ , the quotient group  $\mathbb{H}/\mathbb{Z}$  is simple. We denote by  $d$  the  $F$ -rank of that quotient.

An outline of the construction of the object  $X_H$  is given in [T]. However, in this expository paper we shall nearly say nothing of this construction.

The  $H$ -set  $X_H$  has a rich structure. First it is a metric space on which  $H$  acts via isometries. Endowed with the metric topology,  $X_H$  is locally compact; it is compact (indeed reduced to a single point) if and only if the topological group  $H/\mathbb{Z}(K)$  is compact, that is if  $d = 0$ .

The set  $X_H$  is endowed with a collection of *apartments* which have the structure of a  $d$ -dimensional affine euclidean space. They play the same rôle as charts in differential geometry. More precisely  $X_H$  is obtained by “gluing” these apartments in such a way that the following properties are satisfied:

- (1)  $X_H$  is the union of its apartments,
- (2)  $H$  acts transitively on the set of apartments and if  $h \in H$ , for any apartment  $\mathcal{A}$  the induced map  $\mathcal{A} \rightarrow h.\mathcal{A}$  is an affine isometry,
- (3) for two apartments  $\mathcal{A}_1, \mathcal{A}_2$ , there exists  $h \in H$  such that  $h.\mathcal{A}_1 = \mathcal{A}_2$  and  $h$  fixes  $\mathcal{A}_1 \cap \mathcal{A}_2$  pointwise.

We fix a maximal  $K$ -split torus  $\mathbb{T}$  of  $\mathbb{H}$  and write  $T = \mathbb{T}(K)$ . Let  $N(T)$  be the normalizer of  $T$  in  $H$  and  $T^0$  be the maximal compact subgroup of  $T$ . The groups  $W^\circ = N(T)/T$  and  $W^{\text{Aff}} = N(T)/T^0$  are respectively the spherical and the *extended* affine Weyl groups of  $H$  relative to  $T$ . The group  $W^\circ$  is a finite reflexion group, indeed a Coxeter group (cf. [AB]§2). The group  $W^{\text{Aff}}$  is a Coxeter group if and only if  $\mathbb{H}$  is simply connected as a reductive  $F$ -group. In general it may be written as a semidirect product  $W^{\text{Aff}} = \Omega \rtimes W_0^{\text{Aff}}$ , where  $\Omega$  is an abelian group and  $W_0^{\text{Aff}}$  is a Coxeter group.

The torus  $\mathbb{T}$  gives rise to an apartment  $\mathcal{A}_T$  of  $X_H$  which is stabilized by  $N(T)$ . Moreover  $\mathcal{A}_T$  is naturally the geometric realization of a  $d$ -dimensional simplicial complex acted upon by  $N(T)$  via simplicial automorphisms. The maximal dimensional simplices of  $\mathcal{A}_T$  all have the same dimension  $d$ . They are called *chambers*. The subgroup  $T^0$  acts trivially on  $\mathcal{A}$  so that  $\mathcal{A}$  is equipped with an action of  $W^{\text{Aff}}$ , whence a fortiori of  $W_0^{\text{Aff}}$ . The set of chambers of  $\mathcal{A}_T$  is a principal homogeneous space under the action of  $W_0^{\text{Aff}}$ : for any two chambers  $C_1, C_2$  of  $\mathcal{A}_T$ , there exists a unique element  $w$  of  $W_0^{\text{Aff}}$  such that  $C_2 = wC_1$ .

The simplicial structure of  $\mathcal{A}$  extends in a unique way on the whole  $X_H$  so that  $H$  acts on  $X_H$  via simplicial automorphisms. A simplex of  $X_H$  of dimension  $d - 1$  will be called a *codimension 1 simplex*. Each codimension 1 simplex  $D$  of  $X_H$  is contained in two chambers of  $\mathcal{B}$ , for any apartment  $\mathcal{B}$  containing  $D$ , but is contained in  $q_K + 1$  chambers of  $X_H$ . For instance when  $\mathbb{H}$  is  $\text{GL}(2)$  or  $\text{SL}(2)$ , the apartments are euclidean lines, the facets are edges and the codimension 1 facets are vertices. In fact  $X_H$  is a uniform tree of valency  $q_K + 1$ .

**1.2 Chambers and Iwahori subgroups** So buildings may also be viewed as combinatorial objects obtained by gluing chambers together. Moreover together with properties (1), (2), (3), we have:

- (4) for any two chambers of  $X_H$  there exists an apartment containing them both.

From this point of view, it is useful to introduce another distance on  $X_H$  of combinatorial nature. Two chambers  $C_1$  and  $C_2$  are called *adjacent* if the intersection  $C_1 \cap C_2$  is a codimension 1 simplex. A gallery  $\mathcal{G}$  in  $X_H$  is a sequence  $\mathcal{G} = (C_1, C_2, \dots, C_d)$  of chambers such that, for  $i = 1, \dots, d - 1$ ,  $C_i$  and  $C_{i+1}$  are adjacent. The *length* of  $\mathcal{G}$  is  $d - 1$ . The *combinatorial distance*  $d(C, C')$  between two chambers  $C, C'$  is the length of a minimal gallery  $\mathcal{G} = (C_1, C_2, \dots, C_d)$  connecting  $C$  and  $C'$  (i.e. such that  $C_1 = C$  and  $C_d = C'$ ). In fact if  $C$  and  $C'$  lie in  $\mathcal{A}_T$ , any minimal gallery connecting them is contained in  $\mathcal{A}_T$ . Moreover if  $C' = wC$ , where  $w \in W_0^{\text{Aff}}$  (recall that this  $w$  is unique), then  $d(C, C') = l(w)$ , where  $l : W_0^{\text{Aff}} \rightarrow \mathbb{Z}_{\geq 0}$  is the length function of the Coxeter group  $W_0^{\text{Aff}}$  (e.g. see [AB], Corollary 1.75).

The Bruhat-Tits theory attaches to any chamber  $C$  of  $X_H$  a compact open subgroup  $I_C$  of  $H$ : the *Iwahori subgroup* of  $H$  fixing  $C$ . If  $H_C$  denotes the stabilizer of  $C$  in  $H$ , then  $I_C$  is a normal subgroup of  $H_C^0$ , the maximal compact subgroup of  $H_C$ . When  $\mathbb{H}$  is simply connected, one has  $I_C = H_C^0$ , but the containment  $I_C \subset H_C^0$  is strict in general. If  $C$  is a

chamber of  $\mathcal{A}_T$ , then since  $T^0 \subset I_C$ , the product set  $I_C W^{\text{Aff}} I_C$  has a meaning and we have the *Bruhat-Iwahori decomposition* :

$$H = I_C W^{\text{Aff}} I_C = \bigsqcup_{w \in W^{\text{Aff}}} I_C w I_C .$$

The set  $H^0 = I_C W_0^{\text{Aff}} I_C$  is a subgroup of  $H$ . It is equal to  $H$  when  $\mathbb{H}$  is simply connected. The pair  $(I_C, N)$  is a *B-N pair* in  $H^0$  and, as a simplicial complex,  $X_H$  is the building of this *B-N pair* (cf. [AB] §6).

Fix an apartment  $\mathcal{A}_T$ , attached to a maximal split torus  $T$ , containing  $C$ . As a Coxeter group  $W_0^{\text{Aff}}$  is generated by a finite set of involutions  $S$ . An involution  $s \in S$  acts on the apartment  $\mathcal{A}_T$  as the reflection according to the hyperplan containing a certain codimension 1 subsimplex  $D_s$  of  $C$ . This codimension 1 simplex  $D_s$  has the form  $\{v_0, v_1, \dots, v_d\} \setminus \{v_s\}$ . One says that  $v_s$  is a vertex of *type*  $s$  of  $C$ , and that the *opposit* simplex  $D_s$  has *type*  $s$  as well.

More precisely  $W_0^{\text{Aff}}$  has a presentation of the form

$$W_0^{\text{Aff}} = \langle s \in S ; s^2 = 1, (st)^{m_{st}} = 1, s \neq t \in S \rangle$$

where  $m_{st}$  is an integer  $\geq 2$  or is  $\infty$  when  $st$  has infinite order. The length function  $l$  has the following interpretation. If  $w \in W_0^{\text{Aff}}$ ,  $l(w)$  is the number of involutions in any minimal word on the alphabet  $S$  representing  $w$ .

An important feature of buildings is that they are *labellable* as simplicial complexes. Let  $\Delta_d$  be the standard  $d$ -dimensional simplex. Its vertex set is  $\Delta_d^0 = \{0, 1, \dots, d\}$  and any subset of  $\Delta_d^0$  is allowed to be a simplex. A *labelling* of  $X_H$  is a simplicial map  $\lambda : X_H \rightarrow \Delta_d$  which preserves the dimension of simplices. In other words, the labelling  $\lambda$  attaches a number  $\lambda(s) \in \{0, 1, \dots, d\}$  (a *label*) to any vertex  $s$  of  $X_H$ , in such a way that if  $\{s_0, \dots, s_k\}$  is a simplex, then the labels  $\lambda(s_0), \dots, \lambda(s_k)$  are pairwise distinct.

If  $\mathbb{H}$  is simply connected, then the action of  $H$  preserves the labelling. But this is false in general. In any case the action of  $H^0$  is label-preserving. Let  $g \in H$  and  $C = \{s_0, \dots, s_d\}$  be a chambre of  $X_H$ . We may consider the permutation  $\sigma_{g,C}$  in  $\mathfrak{S}_{d+1}$  given by

$$\sigma_{g,C} = \begin{pmatrix} \lambda(s_0) & \lambda(s_1) & \cdots & \lambda(s_d) \\ \lambda(g.s_0) & \lambda(g.s_1) & \cdots & \lambda(g.s_d) \end{pmatrix}$$

Then the signature of  $\sigma_{g,C}$  does not depend on the choice of  $C$ ; we denote it by  $\epsilon_H(g)$ . The map  $\epsilon_H : H \rightarrow \{\pm 1\}$ ,  $g \mapsto \epsilon_H(g)$  is a quadratic character of  $H$ . It is trivial when  $\mathbb{H}$  is simply connected.

**1.3 Behaviour under field extensions** Now let  $E/F$  a tame quadratic extension of  $p$ -adic fields and  $\mathbb{G}$  be a split reductive  $F$ -algebraic group with irreducible root system. Write  $\sigma$  for the generator of  $\text{Gal}(E/F)$ . Write  $X_F$  for the Bruhat-Tits building of

$\mathbb{G}$  and  $X_E$  for the Bruhat-Tits building of  $\mathbb{G}$  considered as an  $E$ -group. These are  $G_F$ -set and  $G_E$ -set respectively, where we put  $G_F = \mathbb{G}(F)$  and  $G_E = \mathbb{G}(E)$ .

We have a natural action of  $\text{Gal}(E/F)$  on  $X_E$  (cf. [T]). In the simply connected case, the simplest way to construct it is as follows. Since  $\text{Gal}(E/F)$  acts continuously on  $G_E$  it acts on the set of maximal compact subgroups of  $G_E$ . If  $s$  is a vertex of  $X_E$ , there is a unique maximal open subgroup  $K_s$  of  $G_E$  which fixes  $s$ . One defines  $\sigma.s$  to be the unique vertex of  $X_E$  fixed by  $\sigma(K_s)$ . Then the action of  $\sigma$  on the vertex set of  $X_E$  extends in an unique way to an affine action of  $\sigma$  on the whole  $X_E$  : if  $x \in X_E$  lies in a chamber  $C = \{s_0, s_1, \dots, s_d\}$  of  $X_E$ , with barycentric coordinates  $(p_0, p_1, \dots, p_d)$ , one defines  $\sigma.x$  to be the barycenter of the weighted system of points  $\{(\sigma.s_0, p_0), \dots, (\sigma.s_d, p_d)\}$ .

The action of  $\sigma$  on  $X_E$  is affine, isometric and simplicial. Moreover  $\sigma$  permutes the apartments of  $X_E$ . The fixed point set  $X_E^{\text{Gal}(E/F)}$  canonically identifies with  $X_F$  as a  $G_F$ -set. So we may view  $X_F$  as contained in  $X_E$ . This is a convex subset and we may normalize the metrics in such a way that  $X_F$  is a submetric space of  $X_E$ . If  $\mathbb{T}$  is a maximal  $F$ -split torus of  $\mathbb{G}$  then it is a maximal  $E$ -split torus of  $\mathbb{G}$  considered as an  $E$ -group. Then the associate apartments  $\mathcal{A}_{\mathbb{T}(F)} \subset X_F$  and  $\mathcal{A}_{\mathbb{T}(E)} \subset X_E$  coincide. In particular  $X_F$  and  $X_E$  have the same dimension.

If  $E/F$  is unramified, then  $X_F$  is a subsimplicial complex of  $X_E$ . However if  $E/F$  is ramified, the inclusion  $X_F \subset X_E$  is not simplicial. In fact in this case, if  $d$  is the dimension of  $X_F$ , any chamber of  $X_F$  is the union of  $2^d$  chambers of  $X_E$ .

If the extension  $E/F$  is not tame, then one still has an embedding  $X_F \subset X_E$  which is  $G_F$ -equivariant, affine and isometric. The subset  $X_F$  lies in the set  $\text{Gal}(E/F)$ -fixed points in  $X_E$ , but this latter set is strictly larger.

**1.4 The building of  $\text{GL}(n)$**  We now work out the example of  $\mathbb{H} = \text{GL}(n)$ , where  $n \geq 2$  is a fixed integer (references for more reading are [AB]§6.9 and [Ga]§§18, 19). Here the group of  $K$ -points of the connected center is  $Z \simeq K^\times$  and the building  $X_H$  is of dimension  $d = n - 1$ . In fact the groups  $\text{GL}(n)$ ,  $\text{PGL}(n)$  and  $\text{SL}(n)$  have the same semisimple building.

To describe the structure of  $H = \text{GL}(n, K)$  and of its building, one makes it act on the  $K$ -vector space  $V = K^n$ . We describe first the spherical and affine Weyl groups. We denote by  $(e_1, \dots, e_n)$  the standard basis of  $V$ . As the group of rational points of a maximal  $K$ -split torus, one takes the diagonal torus  $T$  formed of those elements in  $G$  that stabilize each line  $L_i = Ke_i$ ,  $i = 1, \dots, n$ . Its normalizer  $N$  is the set of elements permuting the lines  $L_i$ ,  $i = 1, \dots, n$ , i.e. the set of *monomial* matrices<sup>1</sup>. The spherical Weyl group  $W^\circ$  is isomorphic to the symmetric group  $\mathfrak{S}_n$ . In fact  $\mathfrak{S}_n$  embeds in  $\text{GL}(n, K)$  in the traditional way so that  $N(T)$  is the semidirect product  $T \rtimes \mathfrak{S}_n$ .

The group  $T^0$  is the set of diagonal matrices in  $\text{GL}(n, K)$  with coefficients in  $\mathfrak{o}_K^\times$ , the group of units of the ring  $\mathfrak{o}_K$ . Let  $D$  denote the group of diagonal matrices whose diagonal coefficients are powers of  $\varpi_K$ . Then the containment  $D \rtimes \mathfrak{S}_n \subset N(T)$  induces an isomorphism of groups  $D \rtimes \mathfrak{S}_n \simeq N(T)/T^0 = W^{\text{Aff}}$ .

<sup>1</sup> A matrix is called monomial if each row or column exactly contains a single non-zero coefficient.

For  $i = 1, \dots, n-1$ , let  $s_i$  be the element of  $\mathrm{GL}(n, K)$  corresponding to the transposition  $(i \ i+1) \in \mathfrak{S}_n$ . Fix a uniformizer  $\varpi_K$  of  $K$  and write

$$\Pi = \begin{pmatrix} 0 & 1 & 0 & & \cdots & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \ddots & \vdots \\ & & & & & & 0 \\ 0 & 0 & & \cdots & \cdots & 0 & 1 \\ \varpi_K & 0 & & \cdots & \cdots & & 0 \end{pmatrix}$$

We put  $s_0 = \Pi s_1 \Pi^{-1}$ . Then a decomposition  $W^{\mathrm{Aff}} = \Omega \rtimes W_0^{\mathrm{Aff}}$  is given by  $\Omega = \langle \Pi \rangle$ , the group generated by  $\Pi$ , and  $W_0^{\mathrm{Aff}} = \langle s_0, s_1, \dots, s_{n-1} \rangle$ , the group generated by the  $s_i$ ,  $i = 0, \dots, n-1$  (or more precisely the canonical images of these elements in  $N(T)/T^0$ ). The  $s_i$  are involutions and the group  $W_0^{\mathrm{Aff}}$  together with the special subset  $S = \{s_0, s_1, \dots, s_{n-1}\}$  of generators is a Coxeter system. More precisely, we have the presentations:

$$W_0^{\mathrm{Aff}} = \langle s_0, s_1 / s_0^2 = s_1^2 = 1 \rangle, \text{ if } n = 2,$$

$$W_0^{\mathrm{Aff}} = \langle s_0, s_1, \dots, s_{n-1} / s_0^2 = \cdots = s_{n-1}^2 = 1, (s_i s_{i+1})^3 = 1, i = 0, \dots, n-1 \rangle, \text{ if } n \geq 3.$$

Here we have the convention that  $s_n = s_0$ .

Let us now describe the building  $X_n$  of  $\mathrm{GL}(n, K)$ . A lattice in the  $K$ -vector space  $V = K^n$  is a  $\mathfrak{o}_K$ -submodule of the form  $L = \mathfrak{o}_K v_1 + \mathfrak{o}_K v_2 + \cdots + \mathfrak{o}_K v_n$ , where  $(v_1, v_2, \dots, v_n)$  is a  $K$ -basis of  $V$ . Two lattices  $L_1$  and  $L_2$  are said equivalent (or homothetic) if there exists  $\lambda \in K^\times$  such that  $L_2 = \lambda L_1$ . The equivalence class of a lattice  $L$  will be denoted by  $[L]$ . We define a simplicial complex  $\mathfrak{X}_n$  as follows. Its vertex set is the set of equivalence classes of lattices in  $V$ . A collection of  $q+1$  lattices  $[L_0], [L_1], \dots, [L_q]$  defines a  $q$ -simplex of  $\mathfrak{X}_n$  if one can choose the representatives so that

$$L_0 \supsetneq L_1 \supsetneq L_2 \supsetneq \cdots \supsetneq L_{q-1} \supsetneq \mathfrak{p}_K L_0.$$

Then  $\mathfrak{X}$  is obviously equipped with an action of  $\mathrm{GL}(n, K)$  via simplicial automorphisms. One can prove [BT2] that the building  $X_n$ , as a  $\mathrm{GL}(n, K)$ -set, naturally identifies with the geometric realization of  $\mathfrak{X}_n$ .

In this identification, the vertices belonging to the standard apartment  $\mathcal{A}_T$  correspond to the classes  $[L]$ , where  $L$  is a lattice *split* by the canonical basis of  $V$ , that is satisfying:

$$L = \sum_{i=1, \dots, n} L \cap (K e_i)$$

This identification is compatible with the action of  $N(T)$ . Moreover, if  $\mathcal{A}_T^0$  denotes the vertex set of  $\mathcal{A}_T$ , we have a surjective map  $\mathbb{Z}^n \rightarrow \mathcal{A}_T^0$ , given by

$$(m_1, \dots, m_n) \mapsto \left[ \sum_{i=1, \dots, n} \mathfrak{p}^{m_i} e_i \right].$$



This map factors through a bijection:  $\mathbb{Z}^n/\mathbb{Z} \simeq \mathcal{A}_T^0$ , where  $\mathbb{Z}$  embeds in  $\mathbb{Z}^n$  diagonally. As an euclidean space  $\mathcal{A}_T$  is isomorphic to  $\mathbb{R}^n/\mathbb{R}$ , where  $\mathbb{R}$  embeds in  $\mathbb{R}^n$  diagonally. The euclidean structure on  $\mathbb{R}^n/\mathbb{R}$  is given as follows: one first equips  $\mathbb{R}^n$  with its usual euclidean structure that one restricts to  $\mathbb{R}_0^n := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 + \dots + x_n = 0\}$ ; then the quotient  $\mathbb{R}^n/\mathbb{R}$  inherits an euclidean structure via the natural isomorphism of  $\mathbb{R}$ -vector spaces  $\mathbb{R}^n/\mathbb{R} \simeq \mathbb{R}_0^n$ . The action of  $N(T) \simeq T \rtimes \mathfrak{S}_n$  on  $\mathcal{A}_T$  is given by

$$\text{diag}(t_1, \dots, t_n) \cdot P_\sigma \cdot (x_1, \dots, x_n) \bmod \mathbb{R} = (x_{\sigma^{-1}(1)} + v_K(t_1), \dots, x_{\sigma^{-1}(n)} + v_K(t_n)) \bmod \mathbb{R} ,$$

for all diagonal matrices  $\text{diag}(t_1, \dots, t_n) \in T$  and all permutation  $\sigma \in \mathfrak{S}_n$ , where  $P_\sigma \in N(T)$  denotes the permutation matrix attached to  $\sigma$ . The *fundamental chamber* in  $\mathcal{A}_T$  is the  $(n-1)$ -simplex  $C_0 = \{[L_0], \dots, [L_{n-1}]\}$ , where for  $k = 0, \dots, n-1$ ,  $L_k$  is given by

$$L_k = \sum_{i=1, \dots, n-k} \mathfrak{o}_K e_i + \sum_{i=n-k+1, \dots, n} \mathfrak{p}_K e_i .$$

The Iwahori subgroup  $I_0$  fixing  $C_0$  is called the *standard Iwahori subgroup* of  $\text{GL}(n, K)$  it is formed of those matrices in  $\text{GL}(n, \mathfrak{o}_K)$  which are upper triangular modulo  $\mathfrak{p}_K$ . The matrix  $\Pi$  stabilizes the chamber  $C_0$ : if  $i \in \{0, 1, \dots, n-1\}$ , we have  $\Pi.[L_i] = [L_{i+1}]$ , where the index  $i$  is considered *modulo*  $n$ . In fact the stabilizer of  $C_0$  in  $G$  is  $\langle \Pi \rangle \rtimes I_0$ , which is also the normalizer of  $I_0$  in  $G$ .

There is a unique labelling  $\lambda$  on  $X_n$  such that  $\lambda([L_i]) = i$ ,  $i = 0, \dots, n-1$ . It is explicite given as follows (cf. [Ga] §19.3). If  $[L]$  is a vertex of  $X_n$ , choose a representative  $L$  such that  $L \subset L_0$ . Since  $\mathfrak{o}_K$  is a principal ideal domain, the finitely generated torsion  $\mathfrak{o}_K$ -module  $L/L_0$  is isomorphic to

$$\mathfrak{o}_K/\mathfrak{p}_K^{k_1} \oplus \dots \oplus \mathfrak{o}_K/\mathfrak{p}_K^{k_n}$$

for some  $n$ -tuple of integers  $0 \leq k_1 \leq k_2 \leq \dots \leq k_n$ . Then

$$\lambda([L]) = \sum_{i=0, \dots, n} k_i \bmod n .$$

The action of the subgroup  $G_0 = I_0 W_0^{\text{Aff}} I_0$  of  $\text{GL}(n, K)$  preserves the labelling. In fact the maximal subgroup of  $\text{GL}(n, K)$  preserving the labelling is

$$\{ g \in \text{GL}(n, K), \quad n | v_K(\det(g)) \} .$$

The value of the quadratic character  $\epsilon = \epsilon_{\text{GL}(n, K)}$  at  $\Pi$  is the signature of the cycle  $(1 \ 2 \ 3 \ \dots \ n)$ , that is  $(-1)^{n-1}$ . Since  $\text{GL}(n, K)$  is the semidirect product  $\langle \Pi \rangle \rtimes G_0$ , it follows that  $\epsilon$  is given by

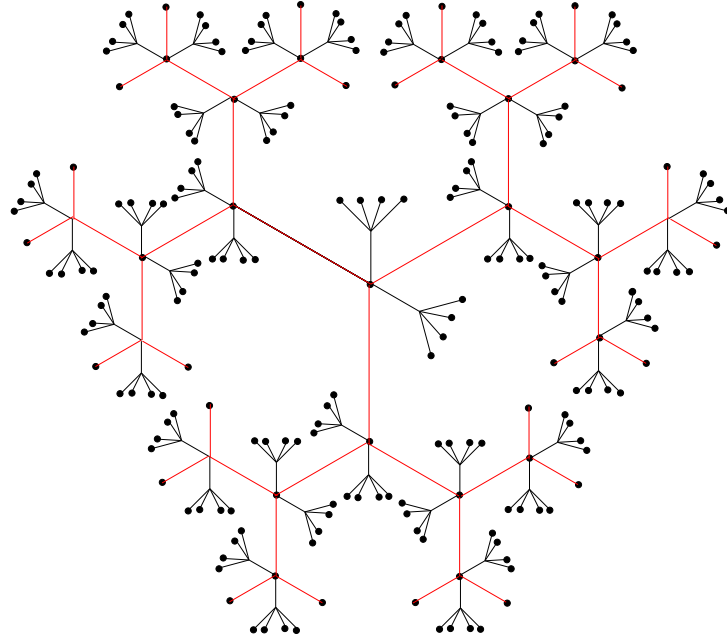
$$\epsilon(g) = (-1)^{(n-1)v_K(\det(g))}, \quad g \in \text{GL}(n, K) .$$

Assume now that  $E/F$  is a quadratic extension of  $p$ -adic fields. Write  $X_F$  and  $X_E$  for the buildings of  $\mathrm{GL}(n, F)$  and  $\mathrm{GL}(n, E)$  respectively. The containment  $X_F \subset X_E$  is given as follows. Set  $V = F^n$  and identify  $\mathrm{GL}(n, F)$  with  $\mathrm{Aut}_F(V)$  and  $\mathrm{GL}(n, E)$  with  $\mathrm{Aut}_E(V \otimes_F E)$ . Then for any  $\mathfrak{o}_F$ -lattice  $L$  of  $V$ , the vertex  $[L]$  of  $X_F$  corresponds to the vertex  $[L \otimes_{\mathfrak{o}_F} \mathfrak{o}_E]$  of  $X_E$ , where  $L \otimes_{\mathfrak{o}_F} \mathfrak{o}_E$  is identified with its canonical image in  $V \otimes_F E$ .

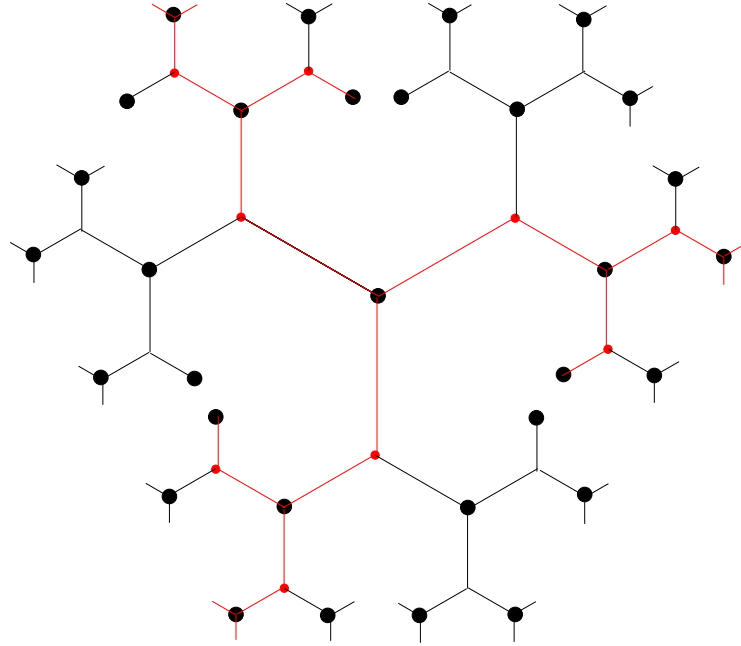
Let  $\mathbb{T}$  be the diagonal torus of  $\mathrm{GL}(n)$ . It corresponds to an apartment  $\mathcal{A}_F$  of  $X_F$  and  $\mathcal{A}_E$  of  $X_E$ . We saw that both apartments identify with  $\mathbb{R}^n/\mathbb{R}$ . Then the containment  $X_F \subset X_E$  restricts to  $\mathcal{A}_F \subset \mathcal{A}_E$  (in fact an equality) where it corresponds to the map

$$\mathbb{R}^n/\mathbb{R} \longrightarrow \mathbb{R}^n/\mathbb{R}, \quad (x_1, \dots, x_n) \bmod \mathbb{R} \mapsto (e(E/F)x_1, \dots, e(E/F)x_n) \bmod \mathbb{R},$$

where  $e(E/F)$  is the ramification index of  $E/F$ .



**Fig. 1.** The embedding  $X_F \subset X_E$ ,  $E/F$  unramified.



**Fig. 2.** The embedding  $X_F \subset X_E$ ,  $E/F$  ramified.

In figure 1, we drew part of the building  $X_E$  when  $F = \mathbb{Q}_2$  and  $E/F$  is unramified. It is an uniform tree of valency 5. The building  $X_F \subset X_E$  is drawn in red; it is an uniform tree of valency 3. Figure 2 represents a part of  $X_E$  when  $F = \mathbb{Q}_2$  and  $E/F$  is ramified. Note that in this case  $E/F$  is not tame. The building of  $X_F$  is drawn in red. Both uniform trees  $X_F$  and  $X_E$  have valency 3. One can see that the embedding  $X_F \subset X_E$  is not simplicial: the red vertices of  $X_E$  are not vertexes of  $X_F$ ; they correspond to middles of chambers.

## 2 Borel-Serre Theorem

**2.1 Statement and ideas of the proof** We fix a  $p$ -adic field  $K$  and a split reductive  $K$ -group  $\mathbb{H}$ . We use the same notation and assumptions than in §1.1. Recall that equipped with its metric topology the affine building  $X_H$  is a locally compact topological space. So for any integer  $k \geq 0$ , we may consider  $H_c^k(X_H, \mathbb{C})$ , the cohomology space with compact support of  $X$  with coefficients in  $\mathbb{C}$ . Here these cohomology spaces are defined by any *reasonable* cohomology theory, e.g. Alexander-Spanier theory [Ma], or cohomology in the sense of sheaf theory [Bre]. In particular the space  $H_c^0(X_H, \mathbb{C})$  is the  $\mathbb{C}$ -vector space of complex locally constant functions with compact support on  $X_H$ . If the dimension  $d$  of the building is  $> 0$ , then  $X_H$  is connected and non-compact, so that  $H_c^0(X_H, \mathbb{C}) = 0$ . From now on, we assume that  $d \geq 1$ .

In [BS], in order to study the cohomology of  $S$ -arithmetic groups, Borel and Serre state and prove the following result.

**Theorem 2.1:** (Cf. [BS] Théorème 5.6 and §5.10) The cohomology space  $H_c^k(X_H, \mathbb{C})$  is trivial when  $k \neq d$ . Moreover as a  $H$ -module  $H_c^d(X_H, \mathbb{C})$  is smooth and irreducible; it is in fact isomorphic to the Steinberg representation of  $H$ .

In fact the topological space  $X_H$  is contractible. Indeed if  $x, y$  are two points of  $X_H$ , and  $t \in [0, 1]$ , the barycenter  $tx + (1 - t)y$  is well defined (it is defined in any apartment  $\mathcal{A}$  containing  $x$  and  $y$  and does not depend on the choice of  $\mathcal{A}$ ). Moreover the map

$$X \times X \times [0, 1] \longrightarrow X, (x, y, t) \mapsto tx + (1 - t)y$$

is continuous. So if  $o$  is any point of  $X$ , the map

$$F : X \times [0, 1] \longrightarrow X, (x, t) \mapsto (1 - t)x + to$$

is a homotopy between  $F(-, 0)$ , the identity map of  $X_H$ , and  $F(-, 1)$  the constant map with value  $o$ .

It follows that the cohomology space (without support)  $H^k(X_H, \mathbb{C})$  are trivial when  $k > 0$ . This also means that if for some  $k > 0$ ,  $H_c^k(X_H, \mathbb{C}) \neq 0$ , this is not due to the existence of “cycles” in  $X_H$  but rather to the fact that  $X_H$  is not compact. So the natural idea that Borel and Serre follow is to compactify the space  $X_H$  by adding a boundary, and this boundary is the Tits building  $Y_H$  of  $H$  topologized in a certain way that we describe now.

The Tits building  $Y_H$  is a simplicial complex. Its vertices are the maximal proper parabolic subgroups of  $H$ . By definition  $r + 1$  such parabolic subgroups  $P_0, P_1, \dots, P_r$  define a  $r$ -simplex if the intersection  $P_0 \cap P_1 \cap \dots \cap P_r$  is a parabolic subgroup of  $H$  (or equivalently contains a Borel subgroup of  $H$ ). Hence the set of simplices of  $Y_H$  is in  $H$ -equivariant bijection with the set of proper parabolic subgroups of  $H$ . The dimension of  $Y_H$  is  $d - 1$ , where  $d$  is the dimension of the affine building  $X_H$ . Fix a simplex of maximal dimension (a chamber)  $D$  of  $Y_H$ . It is a fundamental domain for the action of  $H$  on  $Y_H$ . If  $B$  denotes the Borel subgroup stabilizing  $D$ , one may view  $Y_H$  as a quotient  $(H/B) \times D / \sim$  of  $(H/B) \times D$ . On  $H/B$  we put the  $p$ -adic topology so that it is a compact set (the group of  $K$ -points of a complete projective variety) and on  $D$  we put the euclidean topology (so as a simplex, it is compact). Then the topology on  $Y_H$  is the quotient topology of  $(H/B) \times D / \sim$ . As a quotient of a compact space  $Y_H$  is compact. In particular we have  $H^q(Y_H, \mathbb{C}) = H_c^q(Y_H, \mathbb{C})$ ,  $q \geq 0$ , where  $H^q$  denotes a cohomology space without condition of support.

The Borel-Serre compactification of  $X_H$  is the disjoint union  $\bar{X}_H = X_H \sqcup Y_H$ . We shall not describe the topology of  $\bar{X}_H$ . Let us just say that the induced topology on  $X_H$  (resp.  $Y_H$ ) is the metric topology (resp. the topology we defined in the last paragraph), that  $X_H$  is open and dense in  $\bar{X}_H$ , that  $Y_H$  is closed. Moreover as  $X_H$ , the topological space  $\bar{X}_H$  is contractible. It follows that its reduced cohomology spaces  $\tilde{H}_c^q(\bar{X}_H, \mathbb{C}) = \tilde{H}^q(\bar{X}_H, \mathbb{C})$  are trivial for all  $q$ . Recall that  $\tilde{H}^q(\bar{X}_H, \mathbb{C})$  is defined as follows:  $\tilde{H}^q(\bar{X}_H, \mathbb{C}) = H^q(\bar{X}_H, \mathbb{C})$ , for  $q > 0$  and  $\tilde{H}^0(\bar{X}_H, \mathbb{C}) = H^0(\bar{X}_H, \mathbb{C})/\mathbb{C}$ , where  $\mathbb{C} \subset H^0(\bar{X}_H, \mathbb{C})$  is viewed as the subspace of constant functions.

In [BS], the authors prove the following result.

**Theorem 2.2:** The cohomology space  $H^q(Y_H, \mathbb{C})$  is trivial for  $q < d - 1$ .

Moreover they describe  $\tilde{H}^{d-1}(Y_H, \mathbb{C})$  as a  $H$ -module. To state their result we need to introduce a bit of notation. We fix a maximal split torus  $T$  of  $H$ , denote by  $\Phi = \Phi(H, T)$  the corresponding root system. We fix a Borel subgroup  $B$  containing  $T$  and a basis  $\Delta$  of the set of positive roots in  $\Phi$  relative to  $B$ . We have a 1–1 correspondence  $I \mapsto P_I$  between subsets of  $\Delta$  and standard parabolic subgroups of  $H$  relative to  $B$  (in particular  $P_\emptyset = B$  and  $P_\Delta = G$ ). For  $I \subset \Delta$  we denote by  $\sigma_I$  the representation of  $H$  in  $C^\infty(H/P_I, \mathbb{C})$ , the space of locally constant complex functions on the compact  $H$ -set  $H/P_I$ . The representations  $\sigma_I$ ,  $I \subset \Delta$ , being smooth and of finite length, one may consider the following element of the Grothendieck group of smooth complex representations of  $H$  of finite length:

$$\text{St}_H := \sum_{I \subset \Delta} (-1)^{|I|} \sigma_I .$$

This element of the Grothendieck group is actually an irreducible representation and is called the *Steinberg representation* (we shall give more details on this representation in the next section). By exploiting the combinatorics of the Tits building  $Y_H$ , Borel and Serre prove that we have the isomorphism of  $H$ -modules:

$$\tilde{H}^{d-1}(Y_H, \mathbb{C}) \simeq \text{St}_H . \quad (1)$$

Now the proof of Theorem 2.1 proceeds as follows. The long exact sequence of the pair of topological spaces  $(\bar{X}_H, Y_H)$  writes ([Ma] Theorem 1.6):

$$\longrightarrow H_c^{k-1}(\bar{X}_H, \mathbb{C}) \longrightarrow H_c^{k-1}(Y_H, \mathbb{C}) \longrightarrow H_c^k(X_H, \mathbb{C}) \longrightarrow H_c^k(\bar{X}_H, \mathbb{C}) \longrightarrow \cdots , \quad k \geq 1 . \quad (2)$$

(The case  $k = 0$  was already considered above). If  $k > 1$ , then  $H_c^{k-1}(\bar{X}_H, \mathbb{C})$  and  $H_c^k(\bar{X}_H, \mathbb{C})$  are trivial since  $\bar{X}_H$  is contractible. Hence a piece of the long exact sequence (2) writes:

$$0 \longrightarrow H_c^{k-1}(Y_H, \mathbb{C}) \longrightarrow H_c^k(X_H, \mathbb{C}) \longrightarrow 0, \quad (3)$$

and we obtain the isomorphism of  $H$ -modules:  $H_c^k(X_H, \mathbb{C}) \simeq H_c^{k-1}(Y_H, \mathbb{C}) = \tilde{H}_c^{k-1}(Y_H, \mathbb{C})$ , as required.

If  $k = 1$ , using again the contractility of  $\bar{X}_H$ , we obtain the exact sequence:

$$\mathbb{C} \simeq H_c^0(\bar{X}_H, \mathbb{C}) \xrightarrow{j} H_c^0(Y_H, \mathbb{C}) \longrightarrow H_c^1(X_H, \mathbb{C}) \longrightarrow 0 ,$$

whence the  $H$ -isomorphism:  $H_c^1(X_H, \mathbb{C}) \simeq H_c^0(Y_H, \mathbb{C})/j(H_c^0(\bar{X}_H, \mathbb{C}))$ , where this latter quotient is easily seen to be isomorphic to  $\tilde{H}_c^0(Y_H, \mathbb{C})$ , as required.

**2.2 Sketch of proof for  $\text{GL}(2)$**  As an exercise, we give a *nearly complete*<sup>2</sup> and elementary proof of Borel-Serre theorem for  $H = \text{GL}(2, F)$ .

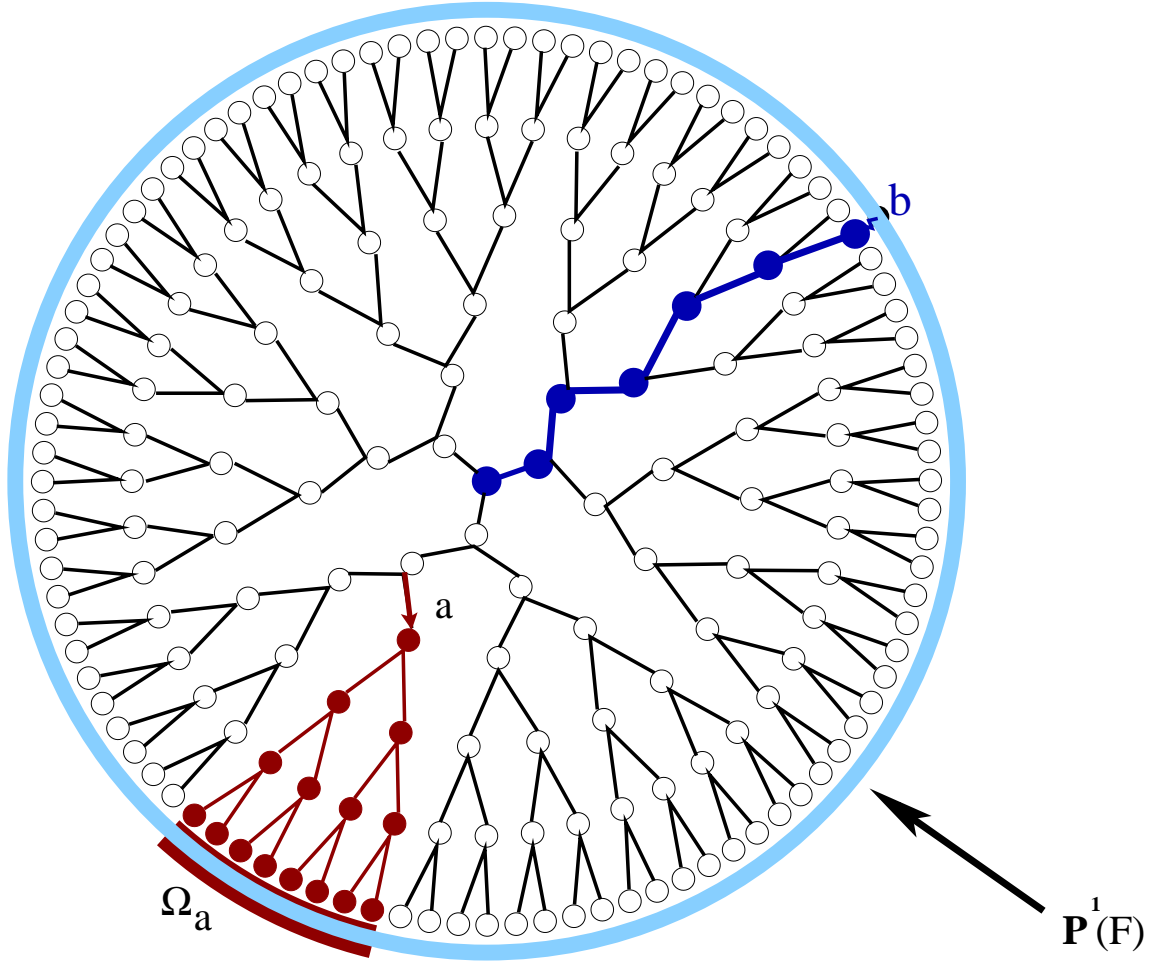
<sup>2</sup> In fact easy results or routine details will be left to the reader

Recall that  $X_H$  is a uniform tree of valency  $q_F + 1$ . The Tits building has dimension  $1 - 1 = 0$ , and as a topological space and  $H$ -set it identifies with the quotient  $H/B$ , where  $B$  is the Borel subgroup of upper triangular matrices. In turn this quotient naturally identifies with the projective line  $\mathbb{P}^1(F)$  (as a topological space and  $H$ -set). Indeed  $H$  acts transitively on the set of lines of  $V = F^2$  and  $B$  is the stabilizer of the line generated by  $(1, 0)$ . So in this very particular case, Borel-Serre theorem claims that we have an isomorphism of  $H$ -modules:  $H_c^1(X_H, \mathbb{C}) \simeq \tilde{H}^0(\mathbb{P}^1(F)) = C^0(\mathbb{P}^1(F))/\mathbb{C}$ , where  $C^0(\mathbb{P}^1(F))/\mathbb{C}$  is the space of locally constant complex functions on  $\mathbb{P}^1(F)$  quotiented by the subspace of constant functions. Of course the  $H$ -module  $C^0(\mathbb{P}^1(F))/\mathbb{C}$  is nothing other than the Steinberg representation of  $H$ .

In the tree case, the Borel-Serre compactification coincides with the compactification obtained by adding *ends* (cf. [Se] I.2.2 and II.1.3, [DT] 1.3.4). A half-geodesic in the tree  $X_H$  is a sequence of vertices  $g = (s_k)_{k \geq 0}$  such that for all  $k \geq 0$ ,  $\{s_k, s_{k+1}\}$  is an edge ( $g$  is a *path*) and  $s_{k+2} \neq s_k$  ( $g$  is *non-backtracking*). Two half-geodesics  $b = (s_k)_{k \geq 0}$  and  $b' = (t_k)_{k \geq 0}$  are said to be *equivalent*, if there exists  $l \in \mathbb{Z}$ , such that  $s_k = t_{k+l}$ , for  $k$  large enough. An *end* in  $X_H$  is an equivalence class of half-geodesics; we denote by  $\text{End}_H$  this set of ends. One observes that  $H$  acts naturally on  $\text{End}_H$  and that, a vertex  $o$  in  $X_H$  being fixed, any end has a unique representative  $(s_k)_{k \geq 0}$  such that  $s_0 = o$ . If  $\{s, t\}$  is an edge of  $X_H$  we define a subset  $\Omega_{(s,t)}$  of  $\text{End}_H$  as follows: an end belongs to  $\Omega_{(s,t)}$  if its representative  $(s_k)_{k \geq 0}$  with  $s_0 = s$  satisfies  $s_1 = t$ . We equip  $\text{End}_H$  with the topology whose basis of open subsets is formed of the  $\Omega_{(s,t)}$ , where  $(s, t)$  runs over the ordered edges of  $X_H$ .

There is a  $H$ -equivariant homeomorphism  $\varphi : \text{End}_H \xrightarrow{\sim} \mathbb{P}^1(F)$  that we describe now. If  $b \in \text{End}_H$ , let  $g = (s_k)_{k \geq 0} \in b$  be the representative satisfying  $s_0 = [\mathfrak{o}_F e_1 + \mathfrak{o}_F e_2]$ , where  $(e_1, e_2)$  is the canonical basis of  $F^2$ . Then (cf. [DT] 1.3.4) one can find a basis  $(v_1, v_2)$  of  $F^2$  such that  $s_k = [\mathfrak{o}_F v_1 + \mathfrak{p}^k v_2]$ ,  $k \geq 0$ . We then define  $\varphi(b)$  to be the line  $Fv_1 \in \mathbb{P}^1(F)$ . Conversely if  $[x : y] := \text{Vect}_F(x, y)$  is a line in  $\mathbb{P}^1(F)$ , one may arrange the representatives  $x, y$  to lie in  $\mathfrak{o}_F$  and to verify:  $x$  or  $y \in \mathfrak{o}_F^\times$ . Then the end  $b$  with representative  $(s_k)$  defined by  $s_k = [\mathfrak{o}_F(x, y) + \mathfrak{p}^k(\mathfrak{o}_F e_1 + \mathfrak{o}_F e_2)]$ ,  $k \geq 0$ , satisfies  $\varphi(b) = [x : y]$ .

In the sequel we canonically identify  $\text{End}_H$  and  $\mathbb{P}^1(F)$  as  $H$ -sets and topological spaces.



**Fig. 3.** An end of  $X_H$  and a basic open subset of  $\mathbb{P}^1(F)$

Since  $X_H$  is a simplicial complex, its cohomology space  $H_c^1(X_H, \mathbb{C})$  may be computed via the complex of simplicial cochains. This argument will be used again later in these notes. Write  $X_H^0$  for the set of vertices of  $X_H$ , and  $X_H^1$  for the set of *oriented edges* of  $X_H$ , that is the set of ordered pairs  $(s, t)$ , where  $\{s, t\}$  is an edge of  $X_H$ . We denote by  $C_c^0(X_H)$  the  $\mathbb{C}$ -vector space of *0-cochains with compact support*, that is the set of functions  $f : X_H^0 \rightarrow \mathbb{C}$  with finite support. Similarly  $C_c^1(X_H)$  denotes the  $\mathbb{C}$ -vector space of *1-cochains with compact support*, that is the set of functions  $\omega : X_H^1 \rightarrow \mathbb{C}$  with finite support and satisfying  $\omega(s, t) = -\omega(t, s)$ , for all edges  $\{s, t\}$  of  $X_H$ . We have a coboundary operator  $d : C_c^0(X_H) \rightarrow C_c^1(X_H)$  given by  $df(s, t) = f(s) - f(t)$ , for all edges  $\{s, t\}$ . The spaces  $C_c^i(X_H)$  are naturally smooth  $H$ -modules and the map  $d$  is  $H$ -equivariant. As a smooth representation of  $H$ , the cohomology of  $X_H$  is given by the cohomology of the complex :

$$0 \longrightarrow C_c^0(X_H) \xrightarrow{d} C_c^1(X_H) \longrightarrow 0 .$$

In particular we have an isomorphism of  $H$ -modules :  $H_c^1(X_H, \mathbb{C}) \simeq C_c^1(X_H)/dC_c^0(X_H)$ .

We are now going to construct a natural  $H$ -equivariant map  $\Psi : C_c^1(X_H) \rightarrow C^0(\mathbb{P}^1(F))/\mathbb{C}$  and prove that it is onto and has kernel  $dC_c^0(X_H)$ . The Borel-Serre theorem will follow.

Once for all fix a vertex  $o \in X_H^0$  (e.g.  $o = [\mathbf{o}_F e_1 + \mathbf{o}_F e_2]$ ). If  $p = (s_0, s_1, \dots, s_k)$  is any path in  $X_H$  and  $\omega \in C_c^1(X_H)$ , define the *integral of  $\omega$  along  $p$*  to be

$$\int_p \omega := \sum_{i=0, \dots, k-1} \omega(s_i, s_{i+1}) .$$

Note that, since  $X_H$  is simply connected,  $\int_p \omega$  only depends on  $\omega$  and the origin and end  $s_0$  and  $s_k$  of  $p$ . If  $\omega \in C_c^1(X_H)$  and  $b$  is an end of  $X_H$  with representative  $(s_k)_{k \geq 0}$  (normalized by  $s_0 = o$ ), the sequence  $(\int_{(s_0, \dots, s_k)} \omega)$  is stationnary since  $\omega$  has finite support; denote by  $\varphi_\omega(b)$  its ultimate value. It is a routine exercice to prove that  $f_b$  is a locally constant function on  $\mathbb{P}^1(F)$ . We then define  $\Psi(b)$  to be the image of the function  $\varphi_\omega : \mathbb{P}^1(F) \rightarrow \mathbb{C}$  in  $C^0(\mathbb{P}^1(F))/\mathbb{C}$ . Note that if one changes the origin vertex  $o$  the function  $\varphi_\omega$  is modified by an additive constant so that its image in  $C^0(\mathbb{P}^1(F))/\mathbb{C}$  does not change.

Let us prove that the kernel of  $\Psi$  is  $dC_c^0(X_H)$ . The containment  $dC_c(X_H) \subset \text{Ker } \Psi$  is easy for if  $f \in C_c^0(X_H)$ , one has  $\varphi_{df}(b) = -f(o)$  for any end  $b$  so that  $\varphi_{df}$  is constant. Let  $\omega \in \text{Ker } \Psi$ . This means that  $\varphi_\omega$  is a constant function. Let  $c$  be its value. Then if an end  $b$  has representative  $(s_k)_{k \geq 0}$ , with  $s_0 = o$ , one has  $\int_{s_0, \dots, s_k} \omega = c$  for  $k$  large enough. Define a function  $f$  on  $X_H^0$  by

$$f(s) = \left( \int_{p : o \rightarrow s} \omega \right) - c , \quad s \in X_H^0$$

where the notation  $p : o \rightarrow s$  means that  $p$  is any path from  $o$  to  $s$ . Since  $X_H$  is a tree,  $f$  is well defined, and by a compactness argument its has finite support. It is finally clear that  $df = \omega$ , as required.

For the surjectivity of  $\Psi$ , fix  $g \in C^0(\mathbb{P}^1(F))$  be any locally constant function. On has to find  $\omega \in C_c^1(X_H)$  satisfying  $\varphi_\omega = g$ . For any integer  $r \geq 1$ , consider the finite subtree  $S(o, r)$  of formed of points at distance less than or equal to  $r$  from a fixed vertex  $o$ .<sup>3</sup> Then  $S(o, r)$  contains

$$1 + (q_F + 1) + (q_F + 1)q_F + \dots + (q_F + 1)q_F^i + \dots + (q_F + 1)q_F^{r-1} = 1 + (q_F + 1) \frac{q_F^r - 1}{q_F - 1}$$

vertices. A vertex of  $S(o, r)$  has valency 1 or  $q_F + 1$  according to whether it is an *end* of  $S(o, r)$  or not. Let  $s_i$ ,  $i = 1, \dots, (q_F + 1)q_F^{r-1}$  be an indexing of the ends of  $S(o, r)$ , and for each  $s_i$ , let  $t_i$  denote the unique neighbour vertex of  $s_i$  in  $S(o, r)$ . Then we have the following partition of  $\mathbb{P}^1(F)$ :

$$\mathbb{P}^1(F) = \bigsqcup_{i=1, \dots, (q_F+1)q_F^{r-1}} \Omega_{(t_i, s_i)} ,$$

<sup>3</sup> The distance on the tree is normalized so that the length of an edge is 1.



and by a compactness argument we may assume, by taking  $r$  large enough, that  $g$  is constant on each  $\Omega_{(t_i, s_i)}$ ; write  $c_i$  for this constant value. Now define a function  $f_r$  on the set of vertices of  $S(o, r)$  by  $f_r(s_i) = c_i$ ,  $i = 1, \dots, (q_F + 1)q_F^{r-1}$ , and by giving arbitrary values to  $f_r(s)$ , for all vertices of  $S(o, r)$  which are not ends. Define  $\omega \in C_c^1(X_F, \mathbb{C})$  by  $\omega(s, t) = 0$ , if the edge  $\{s, t\}$  does not lie in  $S(o, r)$ , and by  $\omega(s, t) = f_r(t) - f_r(s)$  otherwise. It is easy to check that  $\Psi_\omega = g$ , as required.

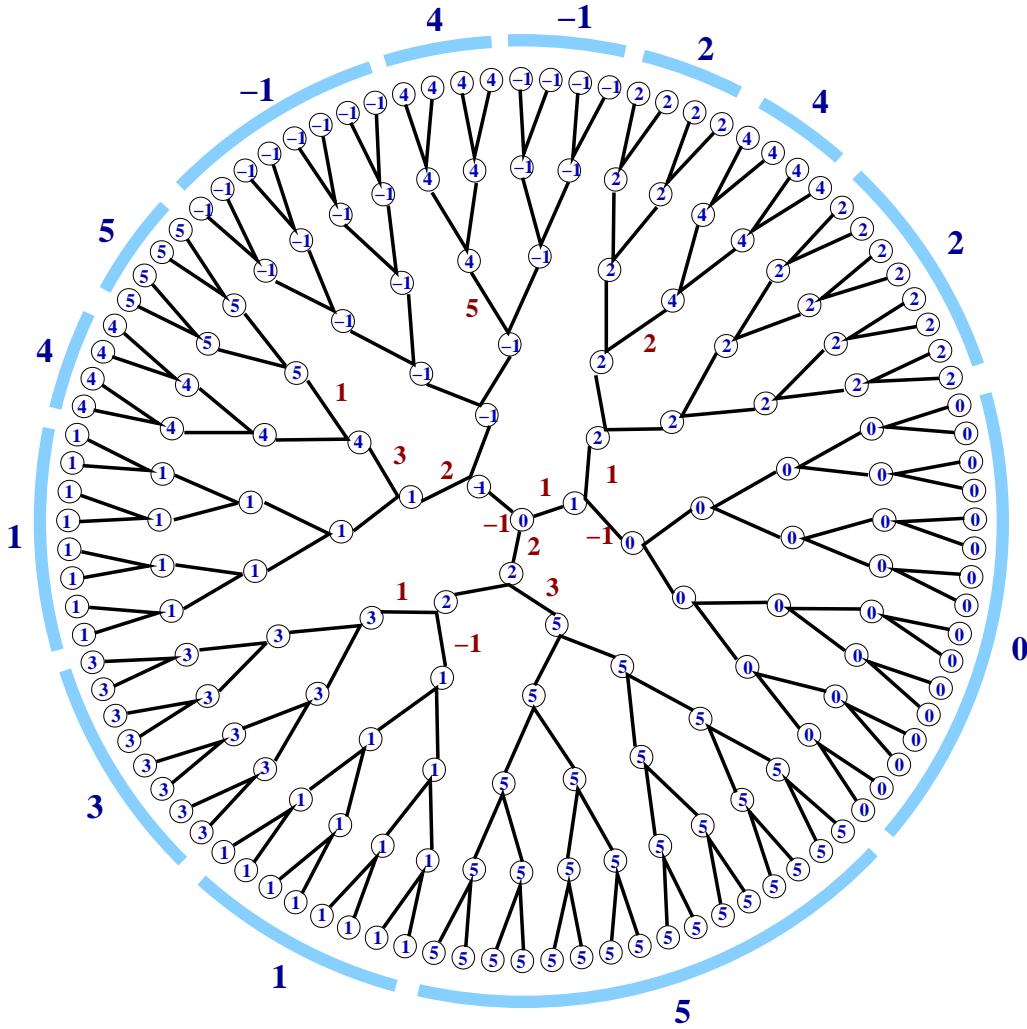


Fig. 4. A 1-cochain  $f$  and its image  $\Psi(f)$ <sup>4</sup>

### 3 Three views of a secret

We give three equivalent constructions of the Steinberg representation. The fact that they are indeed equivalent is a consequence of the Borel-Serre theorem.

<sup>4</sup> Only the non-zero values of  $f$  appear on the figure.

As usual,  $H$  is the group of  $F$ -rational points of a split connected reductive  $F$ -algebraic group  $\mathbb{H}$ .

**3.1 The Steinberg representation via Zelevinski involution** We fix a maximal  $F$ -split torus  $\mathbb{T}$  of  $H$  as well as a Borel subgroup  $\mathbb{B}$  containing  $\mathbb{T}$ . We denote by  $\Phi$  the root system of  $\mathbb{H}$  relative to  $\mathbb{T}$  (that we assume irreducible for simplicity sake), by  $\Phi^+$  the subset of positive roots relative to  $\mathbb{B}$ , and by  $\Delta \subset \Phi^+$  the subset of simple roots. Recall<sup>5</sup> that we have a bijection  $\Theta \mapsto P_\Theta$  between the powerset of  $\Delta$  and the set of parabolic subgroups of  $G$  containing  $B = \mathbb{B}(F)$  (normalized by  $P_\emptyset = B$  and  $P_\Delta = H$ ). Each parabolic  $P_\Theta$  has a standard Levi decomposition  $P_\Theta = M_\Theta U_\Theta$ , where  $U_\Theta$  is (the group of  $F$  rational points of) the unipotent radical of  $P_\Theta$  and  $M_\Theta$  a Levi component.

We denote by  $\mathcal{R}(H)$  the category of smooth complex representations of  $H$ . For  $\Theta \subset \Delta$ ,  $\mathcal{R}(M_\Theta)$  denotes the similar category attached to  $M_\Theta$ , and  $\text{Ind}_{M_\Theta}^H : \mathcal{R}(M_\Theta) \rightarrow \mathcal{R}(H)$ ,  $\text{Res}_H^{P_\Theta} : \mathcal{R}(H) \rightarrow \mathcal{R}(M_\Theta)$  the functors of normalized parabolic induction and normalized restriction (or Jacquet functor) respectively<sup>6</sup>. Both functors take representations of finite length to representations of finite length. In particular if  $\rho$  is an irreducible smooth representation of  $H$ , the representation  $\text{Ind}_{P_\Theta}^G \text{Res}_G^{P_\Theta} \rho$  gives rise to a well defined element of  $[\text{Ind}_{P_\Theta}^G \text{Res}_G^{P_\Theta} \rho]$  of the Grothendieck group  $\mathcal{R}_\#(H)$  of finite length smooth representations of  $H$ .

A.M. Aubert [Au] has generalized the Zelevinski involution, defined by Zelevinsky for  $\text{GL}(N)$ , to the case of any reductive group<sup>7</sup>. For an irreducible smooth representation  $\rho$  of  $H$ , the Aubert-Zelevinski dual of  $\rho$  is the element  $\iota(\rho)$  of  $\mathcal{R}_\#(H)$  defined by

$$\iota(\rho) = \sum_{\Theta \subset \Delta} (-1)^{|\Theta|} \text{Ind}_{P_\Theta}^G \text{Res}_G^{P_\Theta} \rho$$

Then the key result of [Au] is that  $\iota(\rho)$  is, up to a sign, an irreducible representation of  $H$ . We shall denote this latter representation by  $\rho^\iota$ .

**Definition 3.1:** (The Steinberg representation via Zelevinski involution). One defines the Steinberg representation of  $H$  to be the representation  $(\mathbf{1}_H)^\iota$ , that is the representation obtained by applying the Aubert-Zelevinski involution to the trivial representation  $\mathbf{1}_H$  of  $H$ .

It is a corollary of the proof of Borel-Serre theorem that the Steinberg representation of  $H$ , as previously defined, is in fact isomorphic to the top cohomology with compact support of the affine building of  $H$  as an  $H$ -module. More precisely  $(\mathbf{1}_H)^\iota$  is naturally isomorphic to the top reduced cohomology of the topological Tits building of  $H$ .

In the case  $H = \text{GL}(N, F)$ , there is a simpler description of the Steinberg representation in terms of parabolic induction. Take for  $T = \mathbb{T}(F)$  the subgroup of diagonal matrices and

<sup>5</sup> For more details, cf. [Ca] §1

<sup>6</sup> Cf. [Ca] §3 for more details.

<sup>7</sup> Also see [SS] for another construction of this involution.

for  $B$  the Borel subgroup of upper triangular matrices. For  $a \in F$ , denote by  $|a|_F$  absolute value of  $a$  normalized by  $|\varpi_F|_F = \frac{1}{q_F}$  for any uniformizer  $\varpi_F$  of  $F$ . Finally let  $\tau$  be the character of  $T \simeq (F^\times)^N$  given by

$$\tau(t_1, \dots, t_N) = |t_1|_F^{(1-N)/2} |t_2|_F^{(3-N)/2} \otimes \dots \otimes |t_N|_F^{(N-1)/2}.$$

Then the parabolically induced representation  $\text{Ind}_B^H \tau$  has a unique irreducible  $H$ -quotient, which turns out to be the Steinberg representation of  $H$ .

Historically several definition of the Steinberg representation (or of *special representations*) were given. That we give in this section is the definition that Harish-Chandra gave in [HC] §15. In [Ca2] Casselman proved that this representation, as defined by Harish-Chandra, is in fact irreducible.

### 3.2 The Steinberg representation as a space of harmonic cochains

By the Borel-Serre theorem, the Steinberg representation  $\text{St}_H$  of  $H$  is isomorphic to the top cohomology space  $H_c^d(X_H, \mathbb{C})$  as a  $H$ -module. Since  $X_H$  is a simplicial complex, it is a standard result of algebraic topology that the spaces  $H_c^k(X_H, \mathbb{C})$  can be computed by *simplicial methods*. We recall the definition of the cohomological complex of alterned cochains on  $X_H$  whose cohomology computes  $H_c^*(X_H, \mathbb{C})$ .

Let  $q \in \{0, \dots, d\}$ . An *ordered  $q$ -simplex* of  $X_H$  is an ordered sequence  $(s_0, \dots, s_q)$  of vertices of  $X_H$  such that  $\{s_0, \dots, s_q\}$  is a  $q$ -simplex. We denote by  $X_H^{(q)}$  the set of ordered  $q$ -simplices in  $X_H$ . The space  $C_c^q(X_H)$  of alterned  $q$ -cochains on  $X_H$  with finite support is the space of complex valued functions  $f : X_H^{(q)} \rightarrow \mathbb{C}$  satisfying:

- (a)  $f$  has finite support,
- (b)  $f(s_{\tau(0)}, \dots, s_{\tau(q)}) = \text{sgn}(\tau) f(s_0, \dots, s_q)$ , for all ordered  $q$ -simplices  $(s_0, \dots, s_q)$ , all permutations  $\tau$  of the set  $\{0, 1, \dots, q\}$ , where  $\text{sgn}$  denotes the signature of a permutation.

Each  $C_c^q(X_H)$  is endowed with a structure of smooth  $H$ -module via the formula:

$$(h.f)(s_0, \dots, s_q) = f(h^{-1}.s_0, \dots, h^{-1}.s_q), \quad f \in C_c^q(X_H), \quad h \in H, \quad (s_0, \dots, s_q) \in X_H^{(q)}.$$

For  $q = 0, \dots, d-1$ , we defined a coboundary map<sup>8</sup>  $d : C_c^q(X_H) \rightarrow C_c^{q+1}(X_H)$  by

$$(df)(s_0, \dots, s_{q+1}) = \sum_{i=0, \dots, q+1} (-1)^i f(s_0, \dots, \hat{s}_i, \dots, s_{q+1}), \quad f \in C_c^q(X_H), \quad (s_0, \dots, s_{q+1}) \in X_H^{(q+1)},$$

where  $(s_0, \dots, \hat{s}_i, \dots, s_{q+1})$  denotes the ordered  $q$ -simplex  $(s_0, \dots, s_{i-1}, s_{i+1}, \dots, s_{q+1})$ .

We have a cohomological complex of smooth  $H$ -modules:

$$0 \rightarrow C_c^0(X_H) \xrightarrow{d} \dots \xrightarrow{d} C_c^q(X_H) \xrightarrow{d} C_c^{q+1}(X_H) \xrightarrow{d} \dots \xrightarrow{d} C_c^d(X_H) \rightarrow 0.$$

---

<sup>8</sup> The reader will forgive me to use the same symbol  $d$  to denote the dimension of  $X_H$  and the coboundary map.

Finally the  $H$ -modules  $H_c^q(X_H, \mathbb{C})$ ,  $q = 0, \dots, d$ , are given by

$$H_c^q(X_H, \mathbb{C}) = \ker (d : C_c^q(X_H) \longrightarrow C_c^{q+1}(X_H)) / dC_c^{q-1}(X_H)$$

with the convention that  $C_c^i(X_H) = 0$ , for  $i = -1, d+1$ . In particular we have  $\mathbf{St}_H \simeq H_c^d(X_H, \mathbb{C}) = C_c^d(X_H) / dC_c^{d-1}(X_H)$ .

We now use the fact that  $X_E$  is labellable in order to give a simpler model of  $\mathbf{St}_H$ . So fix a labelling  $\lambda$  of  $X_E$  and for  $d$ -simplex  $C = \{s_0, \dots, s_d\}$  of  $X_H$ , let  $C_\lambda$  be the unique ordered simplex  $(t_0, \dots, t_d)$  such that  $\{t_0, \dots, t_d\} = \{s_0, \dots, s_d\}$  and  $\lambda(t_i) = i$ ,  $i = 0, \dots, d$ . Let  $\text{Ch}_H$  be the set of chambers of  $X_H$  and  $\mathbb{C}_c[\text{Ch}_H]$  be the set of complex valued functions on  $\text{Ch}_H$  with finite support. Then we have an isomorphism of  $\mathbb{C}$ -vector spaces :

$$\mathbb{C}_c[\text{Ch}_H] \longrightarrow C_c^d(X_H)$$

given as follows. To any function  $f \in \mathbb{C}_c[\text{Ch}_H]$ , the isomorphism attaches the unique  $d$ -cochain  $\tilde{f}$  satisfying

$$\tilde{f}(C_\lambda) = f(C)$$

for any chamber  $C$  of  $X_H$ . Because in general  $H$  does not preserves the orientation of chambers our isomorphism is not an isomorphism of  $H$ -module in general (this is true if  $\mathbb{H}$  is simply connected). Define a  $H$ -module structure on  $\mathbb{C}_c[\text{Ch}_H]$  by

$$(h.f)(C) = \epsilon_H(h) f(h^{-1}C), \quad C \text{ chamber of } X_H,$$

where  $\epsilon_H$  is the quadratic character of  $H$  defined in §1.3. Write  $\mathbb{C}_c[\text{Ch}_H] \otimes \epsilon_H$  for the  $\mathbb{C}$ -vector space  $\mathbb{C}_c[\text{Ch}_H]$  endowed with the  $H$ -module structure we have just defined. We then have :

**Lemma 3.2:** The map  $f \mapsto \tilde{f}$  induced an isomorphism of smooth  $H$ -modules:  $\mathbb{C}_c[\text{Ch}_H] \otimes \epsilon_H \longrightarrow C_c^d(X_H)$ .

The  $H$ -module  $dC_c^{d-1}(X_H)$  is generated by the functions  $df_D$ , where  $D = \{t_0, \dots, t_{d-1}\}$  runs over the  $(d-1)$ -simplices of  $X_H$ , and where  $f_{(t_0, \dots, t_{d-1})}$  is a  $(d-1)$ -cochain with support the set of ordered  $(d-1)$ -simplices of the form  $(t_{\tau(0)}, \dots, t_{\tau(d-1)})$ , where  $\tau$  is any permutation of  $\{0, \dots, d-1\}$ . It is an easy exercise to show that for all codimension 1 simplex  $D$ , up to a sign, and through the isomorphism  $C_c^d(X_H) \simeq \mathbb{C}_c[\text{Ch}_H]$ ,  $df_D$  is the characteristic function of the set of chambers  $C$  containing  $D$ . So we have proved:

**Proposition 3.3:** As an  $H$ -module,  $\mathbf{St}_H$  is the quotient of  $\mathbb{C}_c[\text{Ch}_H] \otimes \epsilon_H$  by the submodule  $\mathbb{C}_c[\text{Ch}_H]^0$  spanned by the function of the form  $g_D$ ,  $D$  codimension 1 simplex of  $X_H$ , where for all chambers  $C$

$$g_D(C) = \begin{cases} 1 & \text{if } C \supset D \\ 0 & \text{otherwise} \end{cases}$$

Recall that, being given a smooth representation  $(\pi, \mathcal{V})$  of  $H$ , we have two notions of *dual representations*. The *algebraic dual* is the representation  $(\pi^*, \mathcal{V}^*)$  where  $\mathcal{V}^* = \text{Hom}_{\mathbb{C}}(\mathcal{V}, \mathbb{C})$  is the space of linear forms on  $\mathcal{V}$ , and where  $H$  acts by

$$(\pi^*(h)\varphi)(v) = \varphi(\pi(h^{-1})v), \quad h \in H, \quad \varphi \in \mathcal{V}^*, \quad v \in \mathcal{V}.$$

The *smooth dual* or *contragredient* is the sub- $H$ -module  $(\tilde{\pi}, \tilde{\mathcal{V}})$  of  $(\pi^*, \mathcal{V}^*)$  formed of *smooth* linear forms, that is linear forms fixed by an open subgroup of  $H$ . We are going to give very simple models for  $\mathbf{St}^*$  and  $\tilde{\mathbf{St}}_H \subset \mathbf{St}_H^*$ . We shall see in the next section that the representation  $\mathbf{St}_H$  is self-dual. So we shall obtain a simple model for  $\mathbf{St}_H$  as well.

In this aim, observe that there is a perfect pairing  $\langle -, - \rangle$  between the  $H$ -modules  $\mathbb{C}_c[\text{Ch}_H]$  and  $\mathbb{C}[\text{Ch}_H]$ , the space of complex valued functions on  $X_H$  with no condition of support. It is given by

$$\langle f, \omega \rangle = \sum_{C \in \text{Ch}_H} f(C)\omega(C), \quad f \in \mathbb{C}_c[X_H], \quad \omega \in \mathbb{C}[X_H].$$

Define the space of *harmonic  $d$ -cochains* on  $X_H$  as the orthogonal  $\mathcal{H}(X_H)$  of  $\mathbb{C}_c[\text{Ch}_H]^0$  in  $\mathbb{C}[\text{Ch}_H]$  relative to the pairing  $\langle -, - \rangle$ . In other words, an element  $\omega \in \mathbb{C}[\text{Ch}_H]$  is said to be *harmonic* if it satisfies the *harmonicity condition*:

$$\sum_{C \supset D} \omega(C) = 0$$

for all codimension 1 simplex  $D$  of  $X_H$ . Finally define  $\mathcal{H}(X_H)^\infty$  to be the space of smooth harmonic  $d$ -cochains, that is harmonic  $d$ -cochains which are fixed by an open subgroup of  $H$ . As a consequence of the previous proposition we have the following.

**Proposition 3.4:** We have two isomorphisms of  $H$ -modules :

$$\mathbf{St}_H^* = \text{Hom}_{\mathbb{C}}(\mathbf{St}_H, \mathbb{C}) \simeq \mathcal{H}(X_H) \otimes \epsilon_H, \quad \mathbf{St}_H \simeq \tilde{\mathbf{St}}_H \simeq \mathcal{H}(X_H)^\infty \otimes \epsilon_H.$$

Smooth harmonic cochains on  $X_H$  are quite tricky objects. One can for instance prove that the unique smooth harmonic cochain with finite support is the zero cochain. In the next next section, we shall exhibit a non-trivial smooth harmonic cochain: an Iwahori spherical vector in  $\mathcal{H}(X_H)^\infty$ .

**3.3 The Steinberg representation via Type Theory** In this section we assume that the algebraic group  $\mathbb{H}$  is simply connected (e.g.  $\mathbb{SL}_n, \mathbb{Sp}_{2n}$ ). Fix an Iwahori subgroup  $I$  of  $H$  and consider the full subcategory  $\mathcal{R}_I(H)$  of  $\mathcal{R}(H)$  defined as follows: a smooth representation  $(\pi, \mathcal{V})$  of  $H$  in a  $\mathbb{C}$ -vector space  $\mathcal{V}$  is an object of  $\mathcal{R}_I(H)$  if, as a  $H$ -module,  $\mathcal{V}$  is generated by the subset  $\mathcal{V}^I$  of vectors fixed by  $I$ .

By [Bo] this category may be described in terms of parabolic induction. One says that an irreducible smooth representation  $\pi$  of  $H$  belongs to the *unramified principal series* if

there exists an unramified character  $\chi$  of a maximal split torus  $T$  of  $H$  such that  $\pi$  is a subquotient of  $\text{Ind}_B^H \chi$ , for some Borel subgroup  $B$  containing  $T$ . Here *unramified* means that  $\chi$  is trivial on the maximal compact subgroup of  $T$ . Then a representation  $(\pi, \mathcal{V})$  is an object of  $\mathcal{R}_I(H)$  if and only if all irreducible subquotients of  $\pi$  belong to the unramified principal series.

In particular the category  $\mathcal{R}_I(H)$  is stable by the operation of taking subquotient. In the terminology of Bushnell and Kutzko's *theory of types* (cf. [BK] for a foundation of this theory), one says that the pair  $(I, \mathbf{1}_I)$  is a *type* for  $H$ .

Let  $\mu$  be a Haar measure on  $H$  normalized by  $\mu(I) = 1$ . Let  $\mathcal{H}(H)$  be the space of complex locally constant functions on  $H$  with compact support. Let  $\mathcal{H}(H, I)$  be the  $\mathbb{C}$ -vector space of bi- $I$ -invariant complex functions on  $H$  with compact support. Equip  $\mathcal{H}(H)$  and  $\mathcal{H}(H, I)$  with the convolution product:

$$f_1 \star f_2(h) = \int_H f_1(hx) f_2(x^{-1}) d\mu(x) .$$

Then  $\mathcal{H}(H)$  is an associative algebra and  $\mathcal{H}(H, I)$  is a subalgebra with unit  $e_I$ , the characteristic function of  $I$ , called the *Iwahori–Hecke algebra* of  $H$ . In fact  $\mathcal{H}(H, I) = e_I \star \mathcal{H}(H) \star e_I$ . This latter algebra is non commutative if the semisimple rank of  $H$  is  $> 0$ . Recall that it is a basic fact of the theory of smooth representations of  $p$ -adic reductive groups that the categories  $\mathcal{R}(H)$  and  $\mathcal{H}(H) - \text{Mod}$  (the category of *non degenerate*<sup>9</sup> left  $\mathcal{H}(H)$ -modules) are “naturally” isomorphic (cf. [Ca] for more details).

If  $(\pi, \mathcal{V})$  is smooth representation of  $H$ , then  $\mathcal{V}^I$  is naturally a left  $\mathcal{H}(H, I)$ -module. In particular we have a functor  $m_I : \mathcal{R}_I(H) \longrightarrow \mathcal{H}(H, I) - \text{Mod}$ , defined by  $(\pi, \mathcal{V}) \mapsto \mathcal{V}^I$ . Historically the following result is the keystone of Type Theory.

**Theorem 3.5:** (Cf. [Bo], [BK]) The functor  $m_I$  is an equivalence of categories. An inverse  $M_I$  of  $m_I$  is given by

$$M_I : \mathcal{H}(H, I) - \text{Mod} \longrightarrow \mathcal{R}_I(H) , M \mapsto \mathcal{H}(H) \otimes_{\mathcal{H}(H, I)} M .$$

In the theorem the  $H$ -module structure of  $\mathcal{H}(H) \otimes_{\mathcal{H}(H, I)} M$  comes from the action of  $H$  on  $\mathcal{H}(H)$  by left translation.

Fix a maximal split torus  $\mathbb{T}$  of  $\mathbb{H}$  such that the chamber  $C$  fixed by  $I$  lies in the apartment  $\mathcal{A}$  attached to  $\mathbb{T}$ . Recall that we have the Iwahori decomposition

$$H = IW^{\text{Aff}}I = \bigsqcup_{w \in W^{\text{Aff}}} IwI$$

where  $W^{\text{Aff}}$  is the affine Weyl group attache to  $\mathbb{T}$ . Also recall that the Coxeter group  $W^{\text{Aff}}$  is generated by a finite set of involutions  $S$  (attached to the pair  $(C, \mathcal{A})$ ).

It follows from the Bruhat-Iwahori decomposition, that as a  $\mathbb{C}$ -vector space,  $\mathcal{H}(H, I)$  has basis  $(e_w)_{w \in W^{\text{Aff}}}$ , where  $e_w$  is the characteristic function of  $IwI$ . The structure of the algebra  $\mathcal{H}(H, I)$  is well known.

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<sup>9</sup> An  $\mathcal{H}(H)$ -module  $M$  is non degenerate if  $\mathcal{H}(H) \cdot M = M$ .

**Theorem 3.6:** (Iwahori-Matsumoto [IM]) The unital  $\mathbb{C}$ -algebra  $\mathcal{H}(H, I)$  has the following presentation: it is generated by the  $e_s$ ,  $s \in S$ , with the relations

- (R1)  $e_s^2 = (q_F - 1)e_s + q_F e_1$ ,  $s \in S$ ,
- (R2) for all distinct  $s, t$  in  $S$ , we have
  - $(e_s e_t)^r e_s = e_t (e_s e_t)^r$ , if  $m_{st} = 2r + 1$ ,
  - $(e_s e_t)^r = (e_t e_s)^r$ , if  $m_{st} = 2r$ ,

where  $m_{st}$  is the order of  $st \in W^{\text{Aff}}$ .

The quadratic relations (R1) writes  $(e_s + 1)(e_s - q_F) = 0$ ,  $s \in S$ . It follows that the algebra  $\mathcal{H}(H, I)$  admits a unique character  $\chi$  (equivalently a 1-dimensional left module) defined by  $\chi(e_s) = -1$ . This character is known as the *special character* of  $\mathcal{H}(H, I)$ . By the equivalence of categories 3.5,  $\chi$  corresponds to an irreducible smooth representation  $(\pi_\chi, \mathcal{V}_\chi)$  of  $H$ . We are going to prove that this representation is nothing other than the Steinberg representation of  $H$ .

We have  $\mathcal{V}_\chi = \mathcal{H}(H) \otimes_{\mathcal{H}(H, I)} \mathbb{C}$  where  $\mathcal{H}(H, I)$  acts on  $\mathbb{C}$  via the character  $\chi$ . Since  $e_I$  is the unit element of  $\mathcal{H}(H, I)$ , this may be rewritten  $\mathcal{V}_\chi = \mathcal{H}(H) \star e_I \otimes_{\mathcal{H}(H, I)} \mathbb{C}$ . The  $(H, \mathcal{H}(H, I))$ -bimodule  $\mathcal{H}(H) \star e_I$  is the space of locally constant function on  $G$  which are right  $I$ -invariant and have compact support. Since  $I$  is the global stabilizer of a chamber of  $X_H$ , the discrete topological space  $H/I$  is isomorphic to the set of chambers in  $X_H$  as a  $H$ -set; we denote by  $\text{Ch}_H$  this set of chambers. It follows that  $\mathcal{H}(H) \star e_I$  identifies with  $\mathbb{C}_c[\text{Ch}_H]$ , the set of complex valued functions with finite support on  $\text{Ch}_H$ . Under this identification, the left  $H$ -module structure of  $\mathbb{C}_c[\text{Ch}_H]$  is the natural one.

The Bruhat-Iwahori decomposition  $I \backslash H / I \simeq W^{\text{Aff}}$  allows us to classify the relative positions of two chambers of  $X_H$ , that is the orbits of  $H$  in  $\text{Ch}_H \times \text{Ch}_H$ : two chambers  $C_1, C_2$  are in position  $w \in W^{\text{Aff}}$ , denoted by  $C_1 \sim_w C_2$ , if the  $H$ -orbit of  $(C_1, C_2)$  contains  $(C_0, wC_0)$ , where  $C_0$  is the chamber fixed by  $I$ . The following lemma is an excellent exercise left to the reader.

**Lemma 3.7:** Under the natural identification  $\mathcal{H}(H) \star e_I \simeq \mathbb{C}_c[\text{Ch}_H]$ , the right  $\mathcal{H}(H, I)$ -module structure of  $\mathbb{C}_c[\text{Ch}_H]$  is given as follows:

$$f \star e_s(C) = \sum_{C' \sim_s C} f(C') = \sum_{C' \supset C_s, C' \neq C} f(C'), \quad f \in \mathbb{C}_c[\text{Ch}_H], \quad s \in S, \quad C \in \text{Ch}_H,$$

where  $C_s$  denotes the codimension 1 subsimplex of  $C$  of type  $s$ .

By definition the tensor product

$$\mathcal{H} \star e_I \otimes_{\mathcal{H}(H, I)} \mathbb{C} \simeq \mathbb{C}_c[\text{Ch}_H] \otimes_{\mathcal{H}(H, I)} \mathbb{C}$$

is the quotient of  $\mathbb{C}_c[\text{Ch}_H] \otimes_{\mathbb{C}} \mathbb{C} = \mathbb{C}_c[\text{Ch}_H]$  by the subspace generated by the functions  $f \star e_w - \chi(w)f$ , where  $f$  runs over  $\text{Ch}_H$  and  $w$  runs over  $W^{\text{Aff}}$ . Since the  $e_s$ ,  $s \in S$ , generate  $\mathcal{H}(H, I)$  as an algebra, this subspace is also generated by the  $f \star e_s - \chi(s)f = f \star e_s + f$ .

By the previous lemma, this is the space of functions generated by the  $f_D$ ,  $D$  codimension 1 simplex of  $X_H$ , defined by  $f_D(C) = 1$  if  $C \supset D$ ,  $f_D(C) = 0$  otherwise. So this space is nothing other than the space  $\mathbb{C}_c[\text{Ch}_H]^0$  defined in §3.2. The following is now a consequence of Proposition 3.3.

**Proposition 3.8:** The  $H$ -module  $\mathcal{V}_\chi = \mathcal{H}(H) \otimes_{\mathcal{H}(H,I)} \mathbb{C}$  is isomorphic to  $\mathbf{St}_H \simeq H_c^d(X_H, \mathbb{C})$ .

**Corollary 3.9:** a) The Steinberg representation of  $H$  has non-zero fixed vectors under the Iwahori subgroup  $I$ . Moreover  $\mathbf{St}_H^I$  is 1-dimensional.

b) The Steinberg representation is self-dual.

Only b) needs to be proved. The  $\mathcal{H}(H, I)$ -module  $m_I(\tilde{\mathbf{St}}_H) = (\tilde{\mathbf{St}}_H)^I$  is the dual of  $\mathbf{St}_H^I$ , and this latter module is 1-dimension, the algebra  $\mathcal{H}(H, I)$  acting via the character  $\chi$ . Since  $\chi$  has real values, the  $\mathcal{H}(H, I)$ -module  $\mathbf{St}_H^I$  is self-dual. It follows that  $\mathbf{St}_H$  is self dual since  $m_I$  is an equivalence of categories.

A non-zero vector in  $\mathbf{St}_H^I$  is called *Iwahori-spherical*. In the next proposition, we describe the line  $\mathbf{St}_H^I$  in the model  $\mathbf{St}_H \simeq \mathcal{H}(X_H)^\infty$ . This will exhibit a non-trivial element of  $\mathcal{H}(X_H)^\infty$ .

**Proposition 3.10:** Let  $C_0$  denote the chamber fixed by  $I$ . There exists a unique Iwahori-spherical vector  $f_{C_0}$  in  $\mathcal{H}(X_H)^\infty$  satisfying  $f_{C_0}(C_0) = 1$ . It is given by

$$f_{C_0}(C) = \left( \frac{-1}{q_F} \right)^{d(C_0, C)}, \quad C \in \text{Ch}_H.$$

In particular if  $C$  is a chamber of the apartment attached to the torus  $\mathbb{T}$ , we have

$$f_{C_0}(C) = \left( \frac{-1}{q_F} \right)^{l(w)}, \quad \text{if } C = wC_0, \quad w \in W^{\text{Aff}}.$$

Indeed let us first remark that  $f_{C_0}$  is  $I$ -invariant; this is due to the fact that, since the action of  $H$  on  $X_H$  is simplicial, it preserves the distance  $d$  between pairs of chambers. In particular,  $I$  being open,  $f_{C_0}$  is a smooth function. Let us prove that it is harmonic. Let  $D$  be a codimension 1 chamber. We need the following lemma whose proof we shall admit.

**Lemma 3.11:** There exists a unique chamber  $C_1$  in  $X_H$  containing  $D$  and such that the distance  $\delta = d(C_0, C_1)$  is minimal. In particular there exists an integer  $\delta \geq 0$  such that among the  $q_F + 1$  chambers containing  $D$ , one is at distance  $\delta$  from  $C_0$  and the other at distance  $\delta + 1$ .

Let  $C_1$  and  $\delta$  be as in the lemma. We have:

$$\begin{aligned} \sum_{C \supset D} f_{C_0}(C) &= \sum_{C \supset D} \left( \frac{-1}{q_F} \right)^{d(C_0, C)} \\ &= \left( \frac{-1}{q_F} \right)^\delta + q_F \left( \frac{-1}{q_F} \right)^{\delta+1} \\ &= 0 \end{aligned}$$

so that the harmonicity condition at  $D$  holds true.



## 4 Distinction of the Steinberg representation

We fix a Galois symmetric space  $G_E/G_F$  as in the introduction. So  $E/F$  is a Galois quadratic extension of non-archimedean local fields and we have  $G_E = \mathbb{G}(E)$ ,  $G_F = \mathbb{G}(F)$ , where  $\mathbb{G}$  is a connected reductive group defined over  $F$ . In [Pr]§7, assuming that the derived group  $\mathbb{G}^{\text{der}}$  is quasi-split over  $F$ , D. Prasad defines a quadratic character  $\epsilon_{\text{Prasad}}$  of  $G_F$  and makes the following conjecture.

**Conjecture 4.1:** Assume that  $\mathbb{G}^{\text{der}}$  is quasi-split and let  $\mathbf{St}_E$  denote the Steinberg representation of  $G_E$ .

- (1) The intertwining space  $\text{Hom}_{G_F}(\mathbf{St}_E, \epsilon_{\text{Prasad}})$  is 1-dimensional.
- (2) For any character  $\chi$  of  $G_F$  such that  $\chi \neq \epsilon_{\text{Prasad}}$ , we have  $\text{Hom}_{G_F}(\mathbf{St}_E, \chi) = 0$ .

As explained in the introduction, this statement is in fact a particular case of a much more general conjecture of Prasad's which predicts the distinction of an irreducible representation of  $G_E$  in terms of its Galois parameter (that is through the conjectural local Langlands correspondence).

Conjecture 4.1 is proved in [BC] under the following assumptions:

- (H1)  $\mathbb{G}$  is split over  $F$ ,
- (H2) the adjoint group of  $\mathbb{G}$  is simple,
- (H3) the extension  $E/F$  is unramified.

In fact Assumption (H2) can easily be removed as shown in [Cou]. In this section we assume that (H1), (H2), (H3) hold. We give some hints for the proof provided in [BC] and we make it simpler by the use of Poincaré series. In §4.5 we shall say a few words on this extension of this result, extension due to François Courtès, to the case where  $E/F$  is *tamely ramified*.

**4.1 The invariant linear form** As in §1.3, we denote by  $X_F$  (resp.  $X_E$ ) the semisimple Bruhat-Tits building of  $G_F$  (resp. of  $G_E$ ). Since  $E/F$  is unramified, we have a natural embedding  $X_F \subset X_E$  which is simplicial,  $G_F$ -equivariant and  $\text{Gal}(E/F)$  equivariant. In particular the set  $\text{Ch}_F$  of chambers of  $X_F$  is naturally a subset of  $\text{Ch}_E$ , the set of chambers of  $X_E$ .

By Proposition 3.4, the Steinberg representation of  $G_E$  is given by  $\mathbf{St}_E \simeq \mathcal{H}(X_E)^\infty \otimes \epsilon_E$ , where:

- $\mathcal{H}(X_E)$  is, as defined in §3.2, the space of harmonic  $d$ -cochains on  $X_E$  ( $d$  is here the semisimple rank of  $\mathbb{G}$ ), and  $\mathcal{H}(X_E)^\infty$  is the subspace of  $G_E$ -smooth vectors.
- $\epsilon_E = \epsilon_{G_E}$  is the quadratic character of  $G_E$  defined in §1.2.

It turns out [Cou] that the restriction  $\epsilon|_{G_F}$  coincides with Prasad's character  $\epsilon_{\text{Prasad}}$ . It follows that the intertwining space  $\text{Hom}_{G_F}(\mathbf{St}_E, \epsilon_{\text{Prasad}})$  is given by

$$\text{Hom}_{G_F}(\mathcal{H}(X_E)^\infty \otimes \epsilon_E, \epsilon_{\text{Prasad}}) = \text{Hom}_{G_F}(\mathcal{H}(X_E)^\infty, \mathbf{1}),$$

where  $\mathbf{1}$  denotes the trivial character of  $G_F$ .

So in order to prove Conjecture 4.1 in our case, we have to establish:

- (1)  $\dim \operatorname{Hom}_{G_F^{\text{der}}}(\mathcal{H}(X_E)^\infty, \mathbf{1}) \leq 1$ ,
- (2)  $\operatorname{Hom}_{G_F}(\mathcal{H}(X_E)^\infty, \mathbf{1}) \neq 0$ ,

where  $G_F^{\text{der}}$  denotes the derived group of  $G$ .

The proofs of (1) and (2) are quite different in nature. We shall say a few words on the proof of (1) in §4.4 and we refer to [BC] for more details. To prove (2) we have to exhibit a non-zero  $G_F$ -invariant linear form

$$\Lambda : \mathcal{H}(X_E)^\infty \longrightarrow \mathbb{C}.$$

It is quite natural to set

$$\Lambda(f) = \sum_{C \in \operatorname{Ch}_F} f(C), \quad f \in \mathcal{H}(X_E)^\infty \quad (4)$$

since, if  $\Lambda$  is well defined, it is clearly linear and  $G_F$ -equivariant.

Of course we have to prove that for each  $f \in \mathcal{H}(X_E)^\infty$  the sum of (4) converges and that there exists  $f_0 \in \mathcal{H}(X_E)^\infty$  such that  $\Lambda(f_0) \neq 0$  (such a vector  $f_0$  is called a *test vector* for  $\Lambda$ ). More precisely we prove the following.

**Proposition 4.2:** (1) If  $f \in \mathcal{H}(X_E)^\infty$ , then the restriction  $f_{\operatorname{Ch}_F}$  lies in  $L^1(\operatorname{Ch}_F)$ , the space of summable complex functions on  $\operatorname{Ch}_F$ .

(2) We have  $\Lambda(f_{\operatorname{Iwahori}}) \neq 0$ , where  $f_{\operatorname{Iwahori}}$  is the Iwahori-spherical vector relative to some fixed chamber  $C$  of  $X_F$ .

In (4.3) we shall give a proof of this proposition which differs from that of [BC] (and which is much simpler). It relies on a good understanding of the combinatorics of chambers in  $X_F$  thanks to the use of Poincaré series.

**4.2 Combinatorics of chambers** We fix a maximal  $F$ -split torus  $\mathbb{T}$  of  $\mathbb{G}$ . Let  $T = \mathbb{T}(F)$  and  $N_G(T)$  be the normalizer of  $T$  in  $G$ . The spherical Weyl group  $N_G(T)/T$  is denoted  $W^{\text{Sph}}$ . Let  $\mathcal{A}$  be the apartment of  $X_F$  attached to  $T$ ; this is also the (Galois fixed) apartment of  $X_E$  attached to  $\mathbb{T}(E)$ . Fix a chamber  $C$  in  $\mathcal{A}$  and write  $I$  for the Iwahori subgroup of  $G_F$  fixing  $C$ . Let  $W^{\text{Aff}} = N(T)/(T \cap I)$  be the extended affine Weyl group of  $G_F$ . As in §1.1, this Weyl group decomposes as  $W^{\text{Aff}} = \Omega \rtimes W_0^{\text{Aff}}$ , where  $W_0^{\text{Aff}}$  is an affine Coxeter group. We denote by  $l$  the length function on  $W_0^{\text{Aff}}$  attached to the chamber  $C$ .

The group  $G_0 := IW_0^{\text{Aff}}I$  is a (normal) subgroup of  $G$  which acts transitively on  $\operatorname{Ch}_F$ . So we may write the disjoint union decomposition:

$$\operatorname{Ch}_F = \bigsqcup_{w \in W_0^{\text{Aff}}} \bigsqcup_{g \in IwI/I} \{g.C\}, \quad (5)$$

where the fact that the union is indeed disjoint comes from the fact that  $\mathcal{A} \cap \operatorname{Ch}_F$  is a fundamental domain for the action of  $I$  on  $\operatorname{Ch}_F$ . For future calculations, we need a formula for the cardinal of  $IwI/I$ ,  $w \in W_0^{\text{Aff}}$ .

**Proposition 4.3:** ([IM] Prop. 3.2) For  $w \in W_0^{\text{Aff}}$ , we have:

$$|IwI/I| = q_F^{l(w)},$$

where  $q_F$  is the cardinal of the residue field of  $F$ .

Let  $N(d)$  denote the number of chambers in  $\mathcal{A}$  at combinatorial distance  $d$  from  $C$ . By definition the Poincaré series of  $W_0^{\text{Aff}}$  is the generating function

$$P_{W_0^{\text{Aff}}}(X) = \sum_{k \geq 0} N(k)X^k = \sum_{w \in W_0^{\text{Aff}}} X^{l(w)}.$$

A close formula for this Poincaré series is known:

**Theorem 4.4:** ([Bott], [St]) The formal series  $P_{W_0^{\text{Aff}}}$  is a rational function given by

$$P_{W_0^{\text{Aff}}}(X) = \frac{1}{(1-X)^{d-1}} \prod_{i=1}^{d-1} \frac{1-X^{m_i}}{1-X^{m_i-1}}$$

where  $m_1, m_2, \dots, m_{d-1}$  are the exponents of the finite Coxeter group  $W^{\text{Sph}}$  (see [Bou] Chap. V, §6, Définition 2).

In particular, the radius of convergence of  $P_{W_0^{\text{Aff}}}(X)$  is 1 and  $P_{W_0^{\text{Aff}}}$  defines a non-vanishing function on the real open interval  $(-1, 1)$ .

For instance if  $W^{\text{Sph}}$  is of type  $A_l$  (case of  $\text{GL}_{l+1}$  or  $\text{SL}_{l+1}$ ), then we have  $m_i = i$ ,  $i = 1, \dots, l$  (cf. [Bou] Planche I).

**4.3 The Poincaré series trick** We begin by proving that for  $f \in \mathcal{H}(X_E)^\infty$ , the infinite sum (4) defining  $\Lambda(f)$  is absolutely convergent, that is  $f|_{\text{Ch}_F} \in L^1(\text{Ch}_F)$ . For this we first use the fact that if a function  $f : \text{Ch}_E \rightarrow \mathbb{C}$  satisfies the harmonicity condition and is smooth under the action of  $G_E$ , then it decreases in a way described as follows.

**Proposition 4.5:** (Cf. [BC]) Let  $f \in \mathcal{H}(X_E)^\infty$ . There exists a real  $K_f > 0$  such that for all chambre  $D$  of  $X_E$ , we have

$$|f(D)| \leq K_f q_E^{-d(C,D)},$$

where  $q_E = q_F^2$  is the cardinal of the residue field  $k_E$ , and where  $d(C, D)$  denotes the combinatorial distance between chambers of  $X_E$ .

Now, for  $f \in \mathcal{H}(X_E)^\infty$ , using decomposition (5), we may write:

$$\begin{aligned} \sum_{D \in \text{Ch}_F} |f(D)| &\leq K_f \sum_{D \in \text{Ch}_F} q_E^{-d(C,D)} \\ &\leq K_f \sum_{w \in W_0^{\text{Aff}}} \sum_{g \in IwI/I} q_E^{-d(C,gC)} \end{aligned}$$

If  $g \in IwI$  for some  $w \in W_0^{\text{Aff}}$ , we write  $g = i_1 w i_2$ , with  $i_1, i_2 \in I$ , so that

$$d(C, gC) = d(i_1^{-1}C, w i_2 C) = d(C, wC) = l(w)$$

where we used the facts that the distance  $d$  is  $G_E$ -invariant and that  $C$  is fixed by  $I$ . So we obtain:

$$\begin{aligned} \sum_{D \in \text{Ch}_F} |f(D)| &\leq K_f \sum_{w \in W_0^{\text{Aff}}} \sum_{g \in IwI/I} q_E^{-l(w)} \\ &\leq K_f \sum_{w \in W_0^{\text{Aff}}} |IwI/I| q_E^{-l(w)} \\ &\leq K_f \sum_{w \in W_0^{\text{Aff}}} q_F^{l(w)} q_E^{-l(w)} \\ &\leq K_f \sum_{w \in W_0^{\text{Aff}}} \left(\frac{1}{q_F}\right)^{l(w)} \\ &\leq K_f P_{W_0^{\text{Aff}}} \left(\frac{1}{q_F}\right) \end{aligned}$$

where we used the fact that  $|IwI/I| = q_F^{l(w)}$  (Proposition 4.3) and that  $q_E = q_F^2$ . Now since the radius of convergence of the series  $P_{W_0^{\text{Aff}}}$  is 1, we obtain  $P_{W_0^{\text{Aff}}}(\frac{1}{q_F}) < +\infty$  and the sum defining  $\Lambda(f)$  is indeed convergent.

We now prove that  $\Lambda$  is non-zero by computing its value at the Iwahori fixed vector of  $\mathbf{St}_E$  given in §3.3. Recall that it is given by

$$f_{\text{Iwahori}}(D) = \left(\frac{-1}{q_E}\right)^{d(C,D)}, \quad D \in \text{Ch}_E.$$

We have:

$$\begin{aligned} \Lambda(f) &= \sum_{D \in \text{Ch}_F} \left(\frac{-1}{q_E}\right)^{d(C,D)} \\ &= \sum_{w \in W_0^{\text{Aff}}} \sum_{g \in IwI/I} \left(\frac{-1}{q_E}\right)^{d(C,D)} \\ &= \sum_{w \in W_0^{\text{Aff}}} |IwI/I| \left(\frac{-1}{q_F^2}\right)^{l(w)} \\ &= \sum_{w \in W_0^{\text{Aff}}} q_F^{l(w)} \left(\frac{-1}{q_F^2}\right)^{l(w)} \\ &= P_{W_0^{\text{Aff}}} \left(-\frac{1}{q_F}\right) \end{aligned}$$

Since  $P_{W_0^{\text{Aff}}}$  does not vanish on the open interval  $(-1, 1)$ , we have proved the following result.

**Proposition 4.6:** Let  $f_{\text{Iwahori}} \in \mathbf{St}_E$  be a non-zero Iwahori-spherical vector of  $\mathbf{St}_E$  and  $\Lambda \in \text{Hom}_{G_F}(\mathbf{St}_E, \epsilon_{\text{Prasad}})$  be a non zero equivariant linear form. Then  $\Lambda(f_{\text{Iwahori}}) \neq 0$ , and for suitable normalizations of  $f_{\text{Iwahori}}$  and  $\Lambda$ , we have the formula:

$$\Lambda(f_{\text{Iwahori}}) = P_{W_0^{\text{Aff}}}(-\frac{1}{q_F}) = \frac{1}{(1 + \frac{1}{q_F})^{d-1}} \prod_{i=1}^{d-1} \frac{1 - (-\frac{1}{q_F})^{m_i}}{1 - (-\frac{1}{q_F})^{m_i-1}}$$

where  $d$  is the rank of the spherical Weyl group  $W^{\text{Sph}}$  of  $\mathbb{G}$  and  $m_1, \dots, m_d$  the exponents of  $W^{\text{Sph}}$ .

Of course, once one knows that  $\Lambda(f_{\text{Iwahori}}) \neq 0$ , one can always find normalizations so that the previous formula holds. The point is that such normalizations are natural in the model of  $\mathbf{St}_E$  given by smooth harmonic cochains.

**4.4 Multiplicity one** The proof of the multiplicity 1 property, i.e. assertion (1) of §4.1, proceeds as follows. We use the natural  $G_E^{\text{der}}$ -isomorphism  $\text{Hom}_{\mathbb{C}}(\mathbf{St}_E, \mathbf{1}) \simeq \mathcal{H}(X_E)$  so that (1) may be rewritten:

$$(1') \quad \dim \mathcal{H}(X_E)^{G_F^{\text{der}}} \leq 1,$$

where  $\mathcal{H}(X_E)^{G_F^{\text{der}}}$  denotes the  $\mathbb{C}$ -vector space of  $G^{\text{der}}$ -invariant harmonic cochains. Let us fix a chamber  $C_0$  in  $X_F$ . The basic idea is to prove that the map

$$j : \mathcal{H}(X_E)^{G_F^{\text{der}}} \longrightarrow \mathbb{C}, \quad f \mapsto f(C_0)$$

is injective. In this aim, we introduce, for each  $\delta = 0, 1, 2, \dots$ , the set

$$\text{Ch}_E^\delta = \{C \in \text{Ch}_E ; d(C, X_F) = \delta\}$$

where  $d(C, X_F)$  denotes the combinatorial distance of  $C$  to  $X_F$ :

$$d(C, X_F) = \min \{d(C, D) ; D \in X_F\}.$$

In particular  $\text{Ch}_E^0 = \text{Ch}_F$ . Let  $f \in \mathcal{H}(X_F)^{G_F^{\text{der}}}$ . We then prove that for each  $\delta \geq 0$ , the restriction of  $f$  to  $\text{Ch}_E^{\delta+1}$  depends only on the restriction of  $f$  on  $\text{Ch}_E^\delta$ . This follows from the harmonicity condition and from a crucial result on the transitivity of the action of  $G_F^{\text{der}}$  on the set of chambers of  $X_E$  ([BC] Theorem (5.1)<sup>10</sup>). It is now easy to prove by an inductive argument that the cochain  $f$  is known once its values on  $\text{Ch}_E^0 = \text{Ch}_F$  are known. Since  $G_F^{\text{der}}$  acts transitively on  $\text{Ch}_F$ ,  $f$  is known once the value  $f(C_0)$  is known and  $j$  is indeed injective.

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<sup>10</sup> The proof of this theorem is due to François Courtès; see the appendix of [BC].

**4.5 The tamely ramified case** Conjecture 4.1 was proved by François Courtès in the tamely ramified case [Cou2], i.e. when  $E/F$  is tamely ramified. This case is much trickier, mainly because, as we noticed in §1.3, the embedding  $X_F \rightarrow X_E$  is not simplicial: a chamber of  $X_F$  is a union of several chambers of  $X_E$ . However the philosophy of Courtès's approach remains roughly the same:

- (1) he proves the multiplicity one result by using the model  $\mathrm{Hom}_{G_F^{\mathrm{der}}}(\mathbf{St}_E, \mathbf{1}) \simeq \mathcal{H}(X_E)^{G_F^{\mathrm{der}}}$ ,
- (2) he proves distinction by exhibiting a non zero element of  $\mathrm{Hom}_{G_F}(\mathbf{St}_E, \epsilon_{\mathrm{Prasad}})$ .

For step (1), Courtès uses an inductive argument similar to that of 4.4. But a new phenomenon appears : in contrast with the case where  $E/F$  is unramified the support of a non zero element in  $\mathcal{H}(X_E)^{G_F^{\mathrm{der}}}$  may be quite complicate. In order to analyse this support, Courtès introduces the notion of the *anisotropy class* of a chamber.

If  $C$  is a chamber of  $X_E$  then it belongs to some  $\mathrm{Gal}(E/F)$ -stable apartment  $\mathcal{A}$  of  $X_E$  (it is not unique). The apartment  $\mathcal{A}$  is in turn attached to some  $\mathrm{Gal}(E/F)$ -stable maximal  $E$ -split torus  $T$  of  $\mathbb{G}$ . To  $T$  one associates its *anisotropy class*: this is an invariant which describes the “anotropic part” of  $T$  as an  $F$ -torus ( $T$  is not necessarily  $F$ -split). It turns out that this anisotropy class does not depend on the choice of  $T$ ; this is what Courtès takes as a definition of the anisotropy class of  $C$ . Then Courtès considers two cases.

*First case:*  $\mathbb{G}$  is of type  $A_{2n}$ . Write  $\mathrm{Ch}_E^0$  for the set of chambers of  $X_E$  lying in  $X_F$ . Then any invariant non-zero harmonic cochain  $f \in \mathcal{H}(X_E)^{G_F^{\mathrm{der}}}$  is trivial on  $\mathrm{Ch}_E^0$  except on a unique  $G_F^{\mathrm{der}}$ -orbit of chambers  $\mathrm{Ch}_c \subset \mathrm{Ch}_E^0$ . Courtès proves by induction that any  $f \in \mathcal{H}(X_E)^{G_F^{\mathrm{der}}}$  is entirely determined by its restriction to  $\mathrm{Ch}_c$ , and multiplicity one follows.

*Second case:*  $\mathbb{G}$  is not of type  $A_{2n}$ , for some integer  $n$ . Then any  $f \in \mathcal{H}(X_E)^{G_F^{\mathrm{der}}}$  is trivial on the whole of  $\mathrm{Ch}_E^0$  and Courtès has to find a new starting point for his induction argument. It turns out that if  $f \in \mathcal{H}(X_E)^{G_F^{\mathrm{der}}}$  and  $C \in \mathrm{Ch}_E$ , the  $f(C) = 0$  except when  $C$  belongs to a certain anisotropy class of chambers denoted by  $\mathrm{Ch}_a$ . Courtès takes as a starting point of his induction the set  $\mathrm{Ch}_a^0$  of chambers  $C$  of anisotropy class  $a$  containing a  $\mathrm{Gal}(E/F)$ -fixed facet of maximal dimension. He then manages to prove that the restriction map

$$\mathcal{H}(X_E)^{G_F^{\mathrm{der}}} \longrightarrow \left\{ f|_{\mathrm{Ch}_a^0} ; f \in \mathcal{H}(X_E)^{G_F^{\mathrm{der}}} \right\}$$

is injective. He is finally reduced to proving that the space of restrictions

$$\left\{ f|_{\mathrm{Ch}_a^0} ; f \in \mathcal{H}(X_E)^{G_F^{\mathrm{der}}} \right\}$$

is one dimensional. This is quite technical for the set  $\mathrm{Ch}_a^0$  is not a single  $G_F^{\mathrm{der}}$ -orbit in general!

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